32BH Challenge Problem Report 3

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Abstract

In this Challenge Report, we analyze the ideas of conservative vector fields and their corresponding potential functions in the context of understanding divergence and curl. These ideas have a great deal of intertwined relationships between themselves. We will look at various examples and situations where these ideas come together. We will use the combination of these ideas to gain a strong algebraic and geometric picture of these ideas and their relationships.

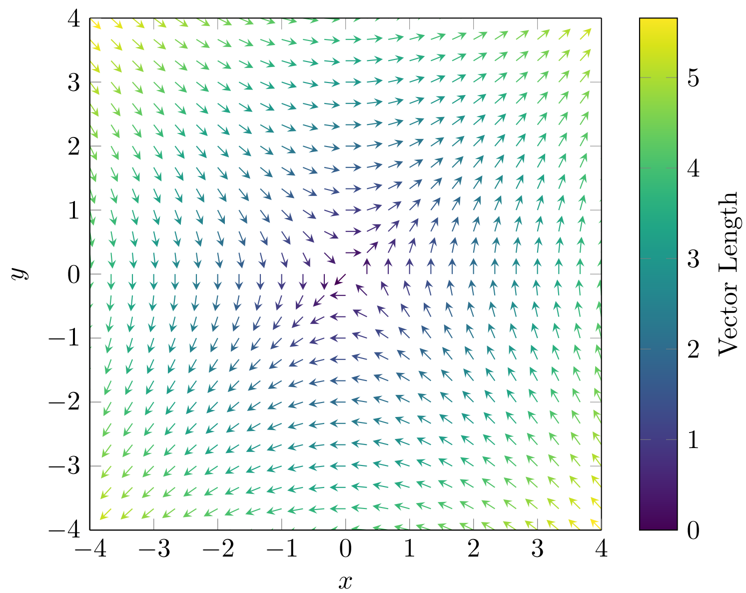
1 Conservative Vector Fields and Potential Functions

In the following section, we will look at an example of a vector field which we will identify as conservative. We will do this by finding the corresponding potential function, comparing that geometrically to the vector field itself. This will provide insight into the geometric interpretation of conservative vector fields and give us a strong foundation to work with them further.

Let's consider the vector field $\boldsymbol{F} = \langle y, x \rangle$

1.1 Graphing the Vector Field

Vector Field $\boldsymbol{F} = \langle y, x \rangle$



In the vector field above, note how the length of vectors are indicated with color to prevent clutter. We now have a picture of how this function $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ behaves. We will now consider the definition of a conservative vector field, looking at the useful tools it offers us in our analysis. It is worth noting that this definition is equivalent to the definition of exact that one may encounter when studying differential forms.

1.2 Finding a Potential Function

Definition 1.1 (Conservative Vector Field). *A vector field* $\boldsymbol{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is called* ***conservative*** *if there is a differentiable function* $f(x_1, \cdots, x_n) : D \to \mathbb{R}$ *such that*

$$\boldsymbol{F} = \nabla f = \langle \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \rangle$$

f *is called a* ***(scalar) potential function*** *for* $\boldsymbol{F}$.

We will now attempt to find a potential function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ for $\boldsymbol{F} = \langle y, x \rangle$ using the definition given. From our definition, the y in our x component of $\boldsymbol{F}$ implies that

$$\frac{\partial f}{\partial x} = y$$

$$\begin{aligned} f(x,y) &= \int y \, dx + \phi(y) \\ &= xy + \phi(y) \end{aligned}$$

Instead of our typical constant $C \in \mathbb{R}$, we can have a pseudoconstant which is really a function of y. This is because the partial derivative of f with respect to x will not consider any excess function of y, and the condition we have essentially just made sure we satisfy will still hold true. Now, let's consider the other piece of information we know from the gradient: $\frac{\partial f}{\partial y}$. We can essentially match up the partial with respect to the $\boldsymbol{F}$ we found with the partial we were given.

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} f(x,y)$$

$$y = y + \phi'(y)$$

$$\implies \phi'(y) = 0$$

Integrating both sides, we can solve for $\phi(y)$

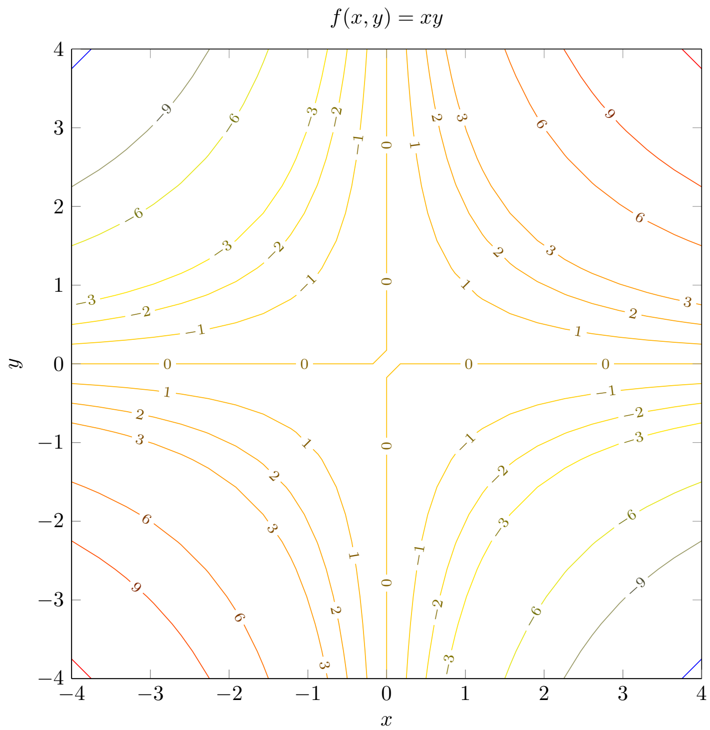
$$\int \phi'(y) \, dy = \int 0 \, dy$$

$$\phi(y) = C \quad C \in \mathbb{R} \text{ (our typical constant)}$$

Thus, plugging back in to $f(x,y) = xy + \phi(y)$, we have our potential function to simply be:

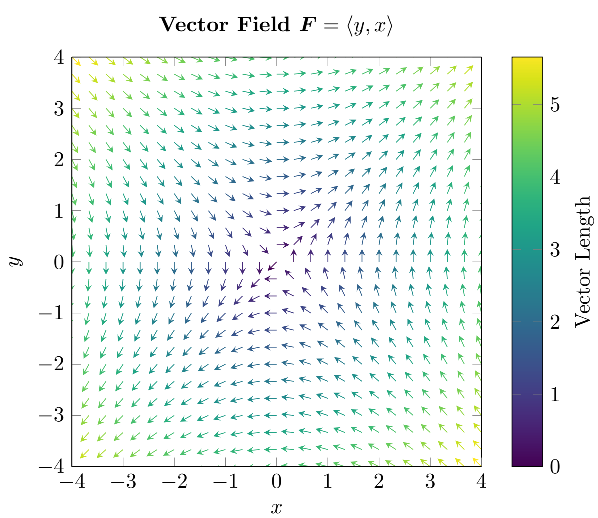
$$f(x,y) = xy + C$$

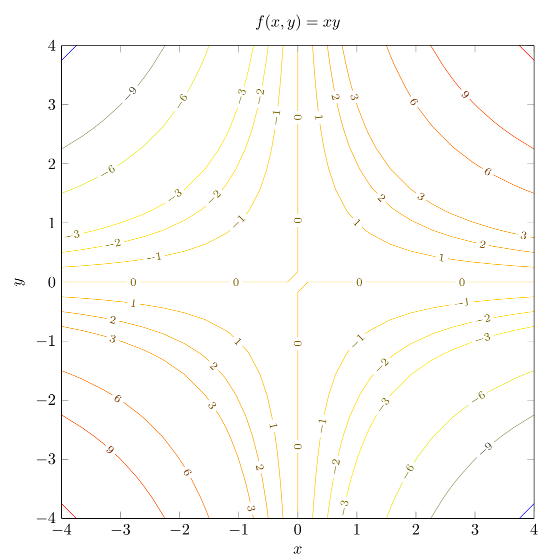
We are simply looking for a singular potential function. We will later observe in Section II that any two potential functions only differ by a constant. As such, let's choose to analyze the $C = 0$ case in our remaining work. Observe the corresponding contour plot.



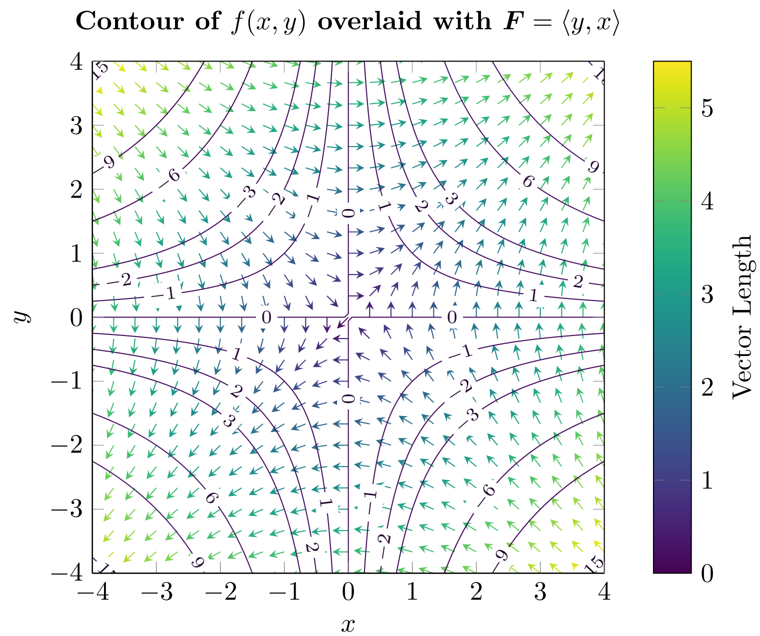
1.3 Comparing Contour Plot and Vector Field

Recall that we have now plotted a vector field, found a potential function, and plotted its contour plot. Let's consider the results our two plots side by side.





We can also look at the two overlaid with one another.



Observe how the vectors of our vector fields are always orthogonal to the level curves of our potential function. This is a key property of the gradient of a function; the gradient will always point orthogonal to the level curves as it must point in the direction of steepest ascent. We will show this with the dot product algebraically. However, it is worth noting this is started in the course lecture notes, "**Proposition 3.1.34.** The gradient ∇f is orthogonal to the (tangent lines of) the level curves." Note, this important distinction: "tangent lines of." While we can generalize this idea to ignore this specification, we will use this idea in showing that two are orthogonal using the dot product.

Consider the parameterization of a level curve $f(x, y) = k$ where $k \in \mathbb{R}$. We -have $\boldsymbol{r}(t) = \langle t, \frac{k}{t} \rangle$ where $t \in \mathbb{R}$. This has a derivative (which is the application of the distinction regarding "tangent line of" the level curve) of $\boldsymbol{r}'(t) = \langle 1, \frac{-k}{t^2} \rangle$. We also have our gradient, $\nabla f = \langle y, x \rangle$. When we plug in the same parameterization for x and y, we have $\nabla f = \langle \frac{k}{t}, t \rangle$. Now, if the dot product of $\boldsymbol{r}'(t)$ and ∇f is 0, then $\boldsymbol{r}(t)$ is orthogonal to the gradient at all t as $\boldsymbol{r}'(t)$ plays the role of being tangent to the curve $\boldsymbol{r}(t)$ at all times. So we have

$$\begin{aligned}
\boldsymbol{r}'(t) \cdot \nabla f &= \langle 1, \frac{-k}{t^2} \rangle \cdot \langle \frac{k}{t}, t \rangle \\
&= \frac{k}{t} - \frac{kt}{t^2} \\
&= \frac{k}{t} - \frac{k}{t} \\
&= 0
\end{aligned}$$

Also, this can clearly be observed in the above graphs. This directly applies here as our original vector field, $\boldsymbol{F} = \langle y, x \rangle$ is the gradient of our potential function $f(x, y) = xy$ by definition.

In the context of our work in this report, this result will be relevant to our later analysis of divergence and curl. We will prove that curl of ∇f is always $\mathbf{0}$ in a later section. For now, however, our result connects our study of vector fields with differential multivariable calculus.

2 Zero Gradient and Constant Functions

In this section, we will attempt to generalize our understanding of the Fundamental Theorem of Calculus and connect it to our understanding of potential functions. Integrating over vector fields, i.e. integrating a function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$, requires a generalized form of the Fundamental Theorem of Calculus.

In order to build up our understanding, let's consider the case where $\nabla f(x, y) = \mathbf{0}$ for all $(x, y) \in D$ where D is an open disk in $\mathbb{R}^2$. We will attempt to show that under these conditions, $f(x, y) : D \to \mathbb{R}$ is a constant function. This is analagous to proving a special case of the following theorem:

Theorem 2.1 (Uniqueness of Potential Functions). *If $\boldsymbol{F}$ is a conservative vector field on an open, connected domain U, then any two potential functions of $\boldsymbol{F}$ differ by a constant.*

Finally, we will prove this theorem for an open disk $D \in \mathbb{R}^2$.

2.1 Showing a function is constant over line segments

Let's consider the following points $P, Q, R \in D$

$$P = (a, b) \quad Q = (c, b) \quad R = (c, d)$$

We will now show that f is constant on the line segments PQ and QR. We can parameterize PQ and QR, which we will call $\boldsymbol{r_1}(t)$ and $\boldsymbol{r_2}(t)$ respectively.

$$\boldsymbol{r_1}(t) = \langle a + t, b \rangle \quad t \in [0, c - a]$$
$$\boldsymbol{r_2}(t) = \langle c, b + t \rangle \quad t \in [0, d - b]$$

We can calculate the derivative of both our parameterizations

$$\boldsymbol{r_1'}(t) = \langle 1, 0 \rangle \quad \boldsymbol{r_2'}(t) = \langle 0, 1 \rangle$$

Consider the vector line integral over each of these straight-line paths in D. Note, we know that these paths stay within D and do not have to concern ourselves with related nuances as D is a known convex subset of $\mathbb{R}^2$, which implies this by definition. Thus, these lines will remain in D no matter the chosen points. However, before we move on, let's attempt to give a proof that the open disk in $\mathbb{R}^2$ is convex, which is what it means for our straight line paths to stay within our disk.

Proof of Convex Nature of Open Disk

We have the definition of a Convex Subset.

Definition 2.2. *A subset $A \subset \mathbb{R}^n$ is said to be a convex subset of $\mathbb{R}^n$ if it contains the line segment joining any two points of A. That is, for all $\boldsymbol{a}, \boldsymbol{b} \in A$, and for all $t \in [0, 1]$, then $\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \in A$.*

Suppose $\boldsymbol{x}, \boldsymbol{y} \in D$, centered at the origin. We can work with $D \subset \mathbb{R}^2$ centered at the origin in this proof and in our related work as we can easily generalize by a translation. In mathematics, it is often useful to work with the simplest case that generalizes completely to our goal–a translation obviously does.

By the definition of our open disk where r is the radius of the disk $D = \{\boldsymbol{z} \in \mathbb{R}^2 : ||\boldsymbol{z}|| < r\}$, we have that

$$||\boldsymbol{x}|| < r \qquad ||\boldsymbol{y}|| < r$$

We want to show that at any point $\boldsymbol{a} = t\boldsymbol{x} + (1 - t)\boldsymbol{y}$ on the line between our two points where $t \in [0, 1]$,

$$||\boldsymbol{a}|| = ||t\boldsymbol{x} + (1 - t)\boldsymbol{y}|| < r$$

Consider

$$\begin{aligned} ||\boldsymbol{a}|| &= ||t\boldsymbol{x} + (1 - t)\boldsymbol{y}|| \\ &\text{by the triangle inequality,} \\ &\leq t||\boldsymbol{x}|| + (1 - t)||\boldsymbol{y}|| \\ &\text{plugging in } r \\ &< tr + (1 - t)r \\ &< r \end{aligned}$$

Thus, we have shown that at any point in the line between arbitrary points in the disk, the points are in the disk as well, since they are of a distance less than the radius of the disk from the center. □

Now that we have shown that our disk is a convex subset of $\mathbb{R}^2$, let us return to the matter at hand. Recall our definition of the vector line integral. We will use this to generalize/apply the fundamental theorem of calculus.

Definition 2.3 (Vector Line Integral). *The* ***line integral of a vector field*** $\boldsymbol{F}$ *along an oriented curve* C *is denoted* $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$*. We define it as the integral of the tangential component of* $\boldsymbol{F}$ *over* C*.*

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} := \int_C (\boldsymbol{F}\cdot\boldsymbol{T})\,ds$$

If we let $\boldsymbol{r}(t)$ *be a positively oriented regular parametrization of an oriented curve* C *for* $a \le t \le b$ *then,*

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\,dt$$

Furthermore, if $\boldsymbol{F} = \langle F_1, F_2, F_3\rangle$*, we have*

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

Our objective is to look at our zero gradient function and use it to show that the corresponding function must be a constant on this disk. Let's consider the vector line integral of ∇f over $\boldsymbol{r_1}$ and then over $\boldsymbol{r_2}$ as well. Using the definition, we have

$$\int_0^{c-a} \nabla f(\boldsymbol{r_1}(t))\cdot\boldsymbol{r_1'}(t)\,dt$$

On one hand, by the chain rule, we have:

$$\int_0^{c-a} \nabla f(\boldsymbol{r_1}(t))\cdot\boldsymbol{r_1'}(t)\,dt = f(\boldsymbol{r_1}(t))|_0^{c-a}$$

This evaluates to

$$f(c,b) - f(a,b) \Leftrightarrow f(Q) - f(P)$$

On the other hand, plugging in our given zero gradient and our calculated $\boldsymbol{r_1}(t)$ gives us

$$\int_0^{c-a} \langle 0,0\rangle\cdot\langle 1,0\rangle\,dt$$

which is simply

$$\int_0^{c-a} 0\,dt = 0$$

Thus, we have that

$$f(Q) - f(P) = 0 \implies f(P) = f(Q)$$

The math in itself gives us that only these two points have equivalent values. However, it is apparent that stopping at any $t \in (0, c-a)$ would produce the same result, implying that f is constant along this straight line path. The exact same result holds true considering f on QR. Observe the same set up

On one hand, by the chain rule, we have:

$$\int_0^{d-b} \nabla f(\boldsymbol{r_2}(t))\cdot\boldsymbol{r_2'}(t)\,dt = f(\boldsymbol{r_2}(t))|_0^{d-b}$$

This evaluates to

$$f(c,d) - f(c,b) \Leftrightarrow f(R) - f(Q)$$

On the other hand, plugging in our given zero gradient and our calculated $\boldsymbol{r_2}(t)$ gives us

$$\int_0^{d-b} \langle 0,0\rangle\cdot\langle 0,1\rangle\,dt$$

which is simply

$$\int_0^{d-b} 0\,dt = 0$$

Thus, we have that

$$f(R) - f(Q) = 0 \implies f(R) = f(Q)$$

Following our same logic as with $\boldsymbol{r_1}(t)$, we have shown that f is constant on the straight line path from QR. Taken together, we have shown that f is constant on any straight line path where either just x or just y varies. We can easily generalize this.

2.2 Constant Function Between Any Two Points

In this section, we want to build on our previous work to show that on our same disk with our same gradient, for any two points $X_1, X_2 \in D$, we have that $f(X_1) = f(X_2)$.

Let's consider the path PQ then along QR, i.e. the two paths we just worked with as a singular path. We can see that along this path, the function remains constant and connects any two arbitrary points. This would imply that for any two points $X_1, X_2 \in D$, we have that $f(X_1) = f(X_2)$. However, we need to be more careful. We need to be sure that our point Q is within D–that our point Q exists as we want it to. We could show this using cases however, we can do this proof more easily generalizing our process from above as opposed to our results from above.

Suppose we have two points, $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ both in D. A parameterized path between them would be

$$\boldsymbol{r}(t) = \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)\rangle \quad t \in [0, 1]$$

It follows that

$$\boldsymbol{r}'(t) = \langle x_2 - x_1, y_2 - y_1\rangle$$

Now, let's consider our same setup from earlier:

On one hand, by the chain rule, we have:

$$\int_0^1 \nabla f(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt = f(\boldsymbol{r}(t))\big|_0^1$$

This evaluates to

$$f(\boldsymbol{r}(1)) - f(\boldsymbol{r}(0)) \Leftrightarrow f(X_2) - f(X_1)$$

On the other hand, plugging in our given zero gradient and our calculated $\boldsymbol{r}(t)$ gives us

$$\int_0^1 \langle 0, 0\rangle \cdot \langle x_2 - x_1, y_2 - y_1\rangle\, dt$$

which is simply

$$\int_0^1 0\, dt = 0$$

Thus, we have

$$f(X_2) - f(X_1) = 0 \implies f(X_1) = f(X_2)$$

Thus, we have generalized our analysis in section 2.1 to show that f is constant anywhere on the open disk when the gradient is zero. We can choose any two points in the disk and they can be connected by a curve which is a constant function. We should generalize this further.

2.3 Generalize to an Arbitrary Conservative Vector Field on an open disk D

In this final part of section 2, we want to show a version of theorem 2.1. Our version states that: *for an arbitrary conservative vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ on an open disk D, then any two potential functions of $\boldsymbol{F}$ differ by a constant.*

Proof:

Suppose we have two potential functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ for a conservative vector field. By definition, we know that $\nabla f_1 = \nabla f_2$. Consider

$$\nabla f_2 - \nabla f_1 = \mathbf{0}$$
$$\nabla(f_2 - f_1) = \mathbf{0} \qquad \text{b/c derivative is linear}$$

We have shown in the previous section that if a function (we are thinking of $f_2 - f_1$ as a single function in this context) has a gradient of zero on an open disk D, it is constant. Because, the $\nabla(f_2 - f_1) = \mathbf{0}$, we have that $f_2 - f_1$ is a constant function. This directly implies that $f_2 = f_1 + C$ where $C \in \mathbb{R}$ is a constant. Therefore, any two arbitrary potential functions for a vector field on our domain (as we chose to arbitrary potential functions to begin with) differ by only a constant. This is a critical step in understanding the discovery and application of potential functions with regard to the powerful tool that are conservative vector fields.

3 Curl and Divergence Introduction and Example

In this section, we will begin to explore the concepts of curl and divergence. Our goal will ultimately be to come up with an example of a vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ that has 0 divergence and $\mathbf{0}$ curl. Let's first build up an understanding of both, so we can easily deduce an example. Consider their definitions

Definition 3.1 (Divergence of a Vector Field). *Given a vector field in* $\mathbb{R}^3$, $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$, *the* ***divergence of F*** *is the scalar-valued function defined by*

$$\text{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Definition 3.2 (Curl of a Vector Field). *Given a vector field in* $\mathbb{R}^3$, $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$, *the* ***curl of F*** *is the vector field defined by*

$$\text{curl}\boldsymbol{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$$

Let's make some observations

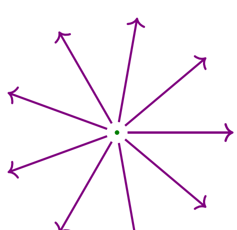
1. Both divergence and curl are defined for vector fields in $\mathbb{R}^3$. While this can be generalized to higher (and lower) dimensions for both concepts (divergence is easier to generalize), we should be looking for an example and to build up understanding in $\mathbb{R}^3$.
2. Both can be written with an abusive (not technically correct) notation with the dot and cross product. They are not technically correct because the dot and cross product are defined for vectors, not operators.
   (a) $\text{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \boldsymbol{F}$
   (b) $\text{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \boldsymbol{F}$

Our first observation gives us an idea of what we should be looking for and a general idea of the primary dimension we should come back to (though we will often be working with a two dimensional analogue). Our second observation is a computational and notational trick but does also give some intuition into the geometric meaning of curl as we will discuss shortly. Let's think about each of divergence and curl geometrically in turn.

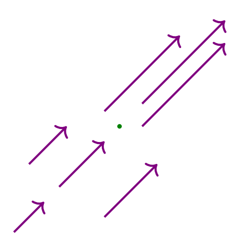
3.1 Geometric Picture of Divergence and Curl

Let's think about divergence geometrically first. Divergence, like it sounds, attempts to measure whether the vector field bunches up or speads out on net at a given point. This concept is called flux. It is perhaps best thought of in a physical context as a fluid flow, whether a fluid tends to converge to a point (negative divergence), diverge away from a point (positive divergence), or is equal on net. These points are thus called sinks, sources, and incompressible points respectively. We will draw some pictures to give a 2d representation of divergence:

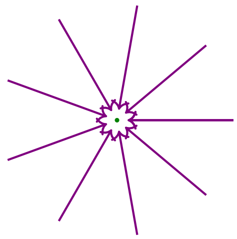
Positive Divergence



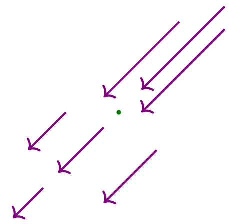
$\text{div}\boldsymbol{F} > 0$



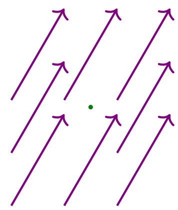
Negative Divergence



$\text{div}\boldsymbol{F} < 0$



Zero Divergence



$\text{div}\boldsymbol{F} = 0$

Some of these graphics were taken and slighty modified from the Challenge Problem Report Document

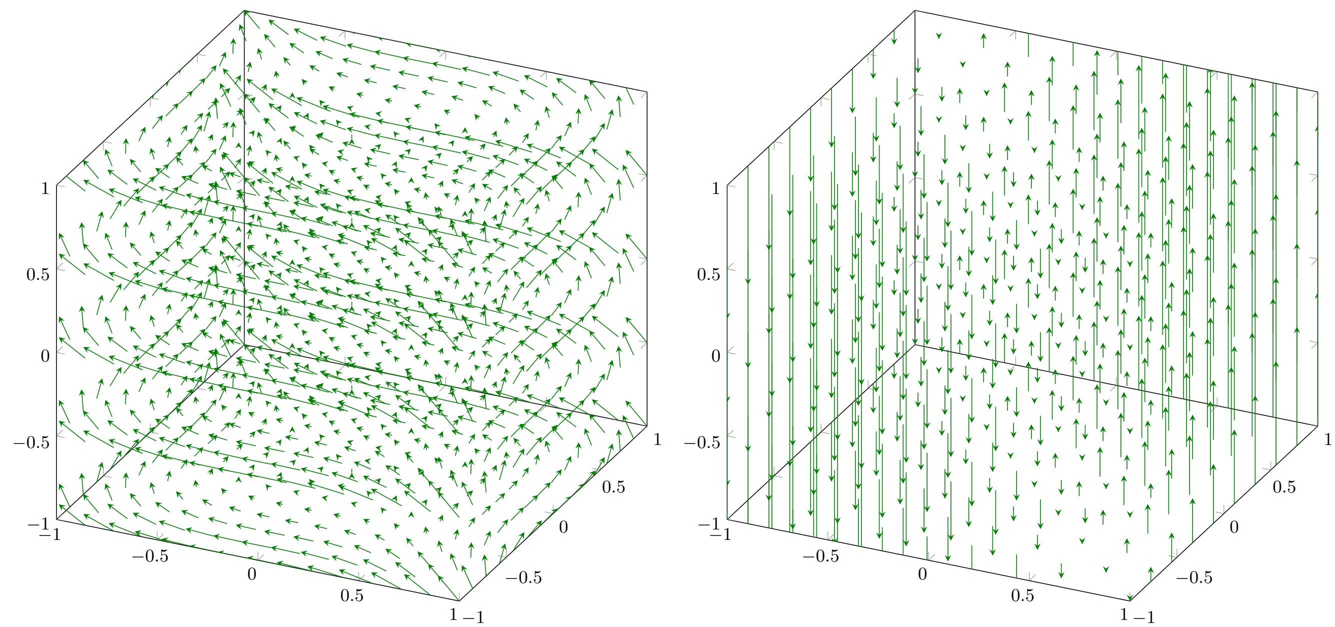
Essentially, from these 2 dimensional pictures, we can observe divergence to be the measure the net passage of the vector field (or perhaps something along the vector field) through in or out of a given point. More specifically, the divergence of a vector field at a point P measures the flux of $\boldsymbol{F}$ through a sphere of radius ε around P. This can be summed up to saying divergence measures the flux at a given point. Again, let's take note of the dimensions before we move on to curl. We defined divergence for vector fields from $\mathbb{R}^3 \to \mathbb{R}^3$. However, it easily generalizes to higher dimensions and to the two dimensional case as we will include below. Nevertheless, our intuition and picture of divergence can remain very similar in dimensions other than $\mathbb{R}^3$.

$$\text{div}\boldsymbol{F} = \sum_i \frac{\partial F_i}{\partial x_i}$$

Now, let's take a moment to think about curl geometrically. We have the following characterization of curl geometrically from the Challenge Problem Report: "Imagine a ball is fixed at a point P in a vector field $\boldsymbol{F}$. Then if the vector field causes the ball to rotate, then the vector $\text{curl}\boldsymbol{F}(P)$ points in the direction of the axis of counterclockwise rotation. Moreover, the angular speed of rotation is directly proportional to the magnitude of $\text{curl}\boldsymbol{F}(P)$."

However, let's take a moment to connect the geometric picture to the formula. This geometric definition is telling us that curl vector points in the direction of the axis of counterclockwise rotation. This is essentially the right hand rule, and it directly connects to our notational trick of using the cross product to define curl.

Let's look at a graph of another example where we can observe magnitude. Note that this code was modified from the challenge report document for this specific function. Consider the vector field $\boldsymbol{F} = \langle -y^2, x^2, 0\rangle$. From our formula, $\text{curl}\boldsymbol{F} = \langle 0, 0, 2x + 2y\rangle$. Let's graph both the vector field on the left and its corresponding curl on the right.



Above, we see an example where a vector field had no z-component and thus had curl exclusively along the z-axis, with a magnitude that depended on the speed a ball would rotate. However, it gives insight into how curl generalizes when said component is non-zero and allows us to get a better picture for how curl behaves. The vectors may be slightly cluttered, but give clear picture of what is primarily going on.

3.2 Non-constant Vector Field with 0 curl and divergence.

Taken together, we now have a strong algebraic and geometric picture of divergence and curl. We now want to find our example of a vector field (non-constant) such that the divergence and curl of each are zero. Let's build on the two vector fields we have looked at in section 1 and section 3.1. We should make note of the following theorem when we come up with our example, but we will not use it or prove it immediately. That will be the subject of the following section. It should just provide insight into what we are looking for.

Theorem 3.3 (Curl of conservative vector field). *If $\boldsymbol{F} = \nabla f$ is a conservative vector field in $\mathbb{R}^3$ with potential function f, then*

$$\text{curl}(\nabla f) = \mathbf{0}$$

Equivalently, $\boldsymbol{F} = \langle F_1, F_2, F_3\rangle$ satisfies the cross-partials condition:

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Thus, we are looking for a conservative vector field with no divergence. We saw both geometrically and algebraically that if we choose vector field such that for each component, it doesn't depend on its corresponding variable, it will have zero divergence. Because we are looking for a conservative vector field, let's choose a potential function that might produce this zero divergence condition and take the gradient.

Consider, $f : \mathbb{R}^3 \to \mathbb{R}$, where $f(x, y, z) = xyz$. Now, let's **define our example vector field** as asked for in the Challenge Report.

$$\boldsymbol{F} = \nabla f = \langle yz, xz, xy\rangle$$

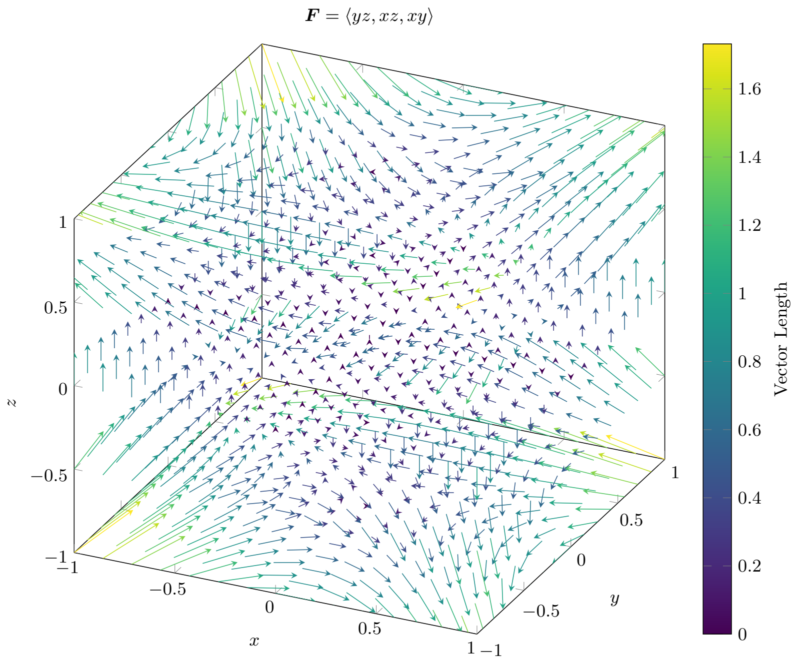
Let's check to see if $\boldsymbol{F}$ has zero divergence and curl. We know by theorem 3.3 that $\boldsymbol{F}$ will have zero curl because it is conservative by definition 1.1. However, we have not introduced this idea in detail or proved it. Instead, let's check curl by plugging into the formula first.

$$\begin{aligned}\text{curl}\boldsymbol{F} &= \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\rangle \\ &= \langle x - x, y - y, z - z\rangle \\ &= \langle 0, 0, 0\rangle \\ &= \mathbf{0}\end{aligned}$$

Now that we have shown curl is zero, checking divergence we see

$$\text{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 0 = 0$$

As a geometric check our work, we will graph our vector field $\boldsymbol{F}$ of which we can hopefully observe has zero divergence and zero curl. While I do not have the technical abilities to graph zoomed in, more detailed fields in LaTeXwith the TikZ package, this should be good enough to give an idea if you imagine zooming into a given neighbourhood.



Once again, while this plot is very cluttered, if you zoom in on any given point, it should be apparent that the divergence is zero. It should be visible that at any given point, the vectors flowing in are equivalent in magnitude/cancel in the proper directions such that the point is incompressible, i.e. the concentration of something flowing along it remains constant. Furthermore, it should be visible that no ball would be rotated by this vector field in a given neighborhood either.

As a whole, this section gave a geometric and algebraic interpretation of divergence and curl and points to the usefulness of conservative function. We have strong tools to model the flow along vector fields and will take our knowledge to prove other relationships between gradients, divergence, and curl in our following sections.

4 Curl of the Gradient

In this section, we will use the tools we have built up to prove that the curl of the gradient of a smooth function is zero. Essentially, we are proving theorem 3.3. In mathematical notation, we want to show that

$$\text{curl}(\nabla f) = \mathbf{0}$$

4.1 Algebraic Proof

Let's do this algebraically first and then consider the geometric picture. Recall that curl is defined for $\mathbb{R}^3$, so $f : \mathbb{R}^3 \to \mathbb{R}$.

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

From our formula for curl, we see that we solely need to satisfy the conditions of the cross partial derivatives outlined in theorem 3.3 as follows

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

This translates to proving the following when we plug f into this formula

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \qquad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

From differential multivariable calculus, we have the following theorem relating taking partial derivatives in different orders.

Theorem 4.1 (Clairaut's Theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $D_i f$, $D_j f$, and $D_i D_j f$ exist and are continuous on an open disk $D \subset \mathbb{R}^n$. Then $D_j D_i f$ exists on D, and moreover, $D_i D_j f = D_j D_i f$ on the disk D.*

From this theorem, we satisfy the domain conditions as our smooth functions will have continuous second order derivatives with the domain of all of $\mathbb{R}^2$. We thus have that all of the necessary conditions to be true to know that

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \qquad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

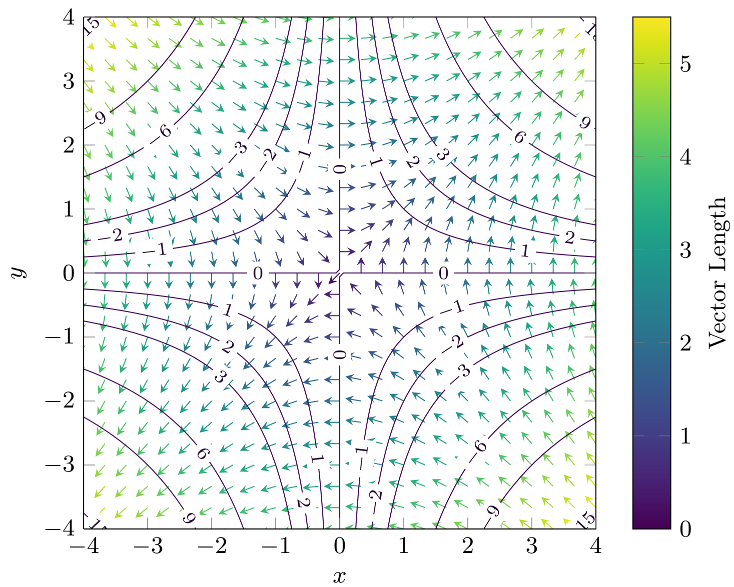
This statement–given very neatly to us by Clairaut's Theorem–is exactly what we wanted to show and thus $\text{curl}(\nabla f) = \mathbf{0}$

□

4.2 Geometric Picture

Let's take a moment to look at the geometric picture. The curl of the gradient being zero means that the vector field that is defined by the gradient does not produce any rotation at any point, with a kind of balanced out push on all sides. This implies directly that the curl of a conservative function as defined in definition 1.1 is always zero. If we choose a simple function and its corresponding conservative function gradient like the one from section 1, we can look see this zero curl vector. Let's look again at $f(x, y) = xy$ and its level curves overlaid with the vector field of its gradient $\nabla f = \boldsymbol{F} = \langle y, x \rangle$. This can correspond to a function $g(x, y, z) = xy$ and its gradient $\nabla g = \boldsymbol{G} = \langle y, x, 0 \rangle$. We are doing the equivalent of looking at a trace of $\boldsymbol{G}$. Note, we are again being careful with how curl is defined. We are using a two dimensional analogue as a powerful tool to analyze sections of the three dimensional case for which curl is well defined.

Contour of $f(x, y)$ overlaid with $\boldsymbol{F} = \langle y, x \rangle$



We can perhaps have an intution that in any given neighborhood, the magnitude of the surrounding vectors and their corresponding angles balances out any rotation, leaving no curl in their wake. Another way to understand this is to connect to the orthogonal relationship between the gradient and contour map. Because the gradient is always orthogonal to the contour map and the curl measures the rotation inward and around, the gradient will not contribute to the curl. While it is difficult to show a visual, more elaborate geometric picture, our algebraic proof and geometric interpretation pushes forward our understanding of curl and its relationship with conservative functions. Let's consider another relationship between divergence and curl in the following section.

5 Divergence of Curl

In this section, we will look at the relationship between the curl of a vector field $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$ and the divergence of its curl. In mathematical notation, we want to show that

$$\text{div}(\text{curl}\boldsymbol{F}) = 0$$

5.1 Algebraic proof

Similar to section four, let's consider our definitions and follow the algebra to complete our proof. Recall from definition 3.2 that the curl of $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\text{curl}(\boldsymbol{F}) = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$$

Thus, we now want to calculate the divergence, the formula for which we can recall from definition 3.1 where $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\text{div}(\boldsymbol{G}) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}$$

Combining these formulas, using the linearity of the derivative, and using Clairaut's Theorem (whose conditions we satisfy because of the properties of smooth functions), we have

$$\begin{aligned}
\text{div}(\text{curl}\boldsymbol{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\
&= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y}\right) \\
&= 0 + 0 + 0 \quad \text{(By Clairaut's Theorem)} \\
&= 0
\end{aligned}$$

Thus, we have shown $\text{div}(\text{curl}\boldsymbol{F}) = 0$ □

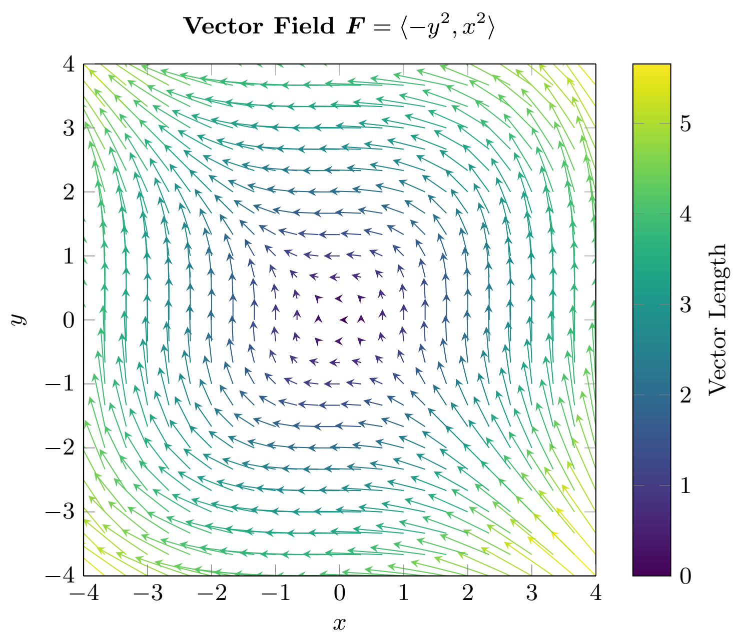
5.2 Geometric Interpretation

Geometrically, this means that the vector field defined by the curl of arbitrary vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is incompressible. At any point, on the vector field, suppose there is a fluid (air) flowing along the field. The density of the air will be constant as it flows, not bunching up or spreading out an any point. Let's think about why. We will reference another three dimensional (two dimensional analogue) example that we considered earlier.

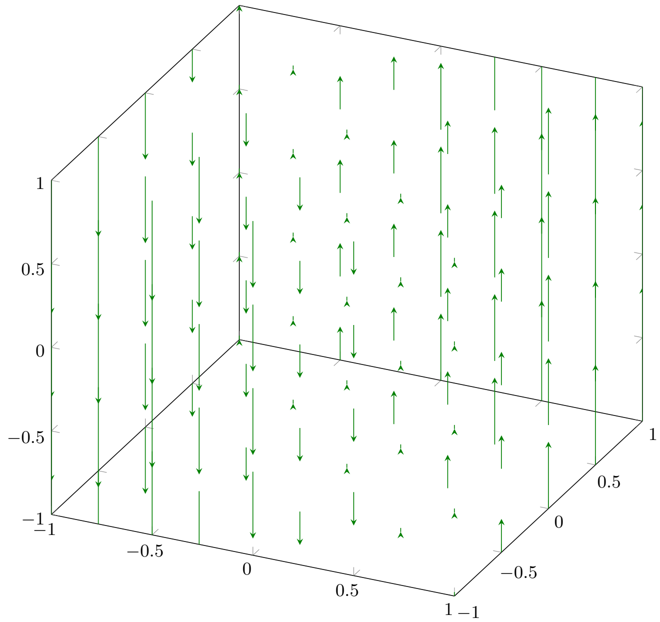
$$\boldsymbol{F} = \langle -y^2, x^2, 0 \rangle$$

From our formula, $\text{curl}\boldsymbol{F} = \langle 0, 0, 2x + 2y \rangle$

Let's look at the 0 trace of this vector field, i.e. the two dimensional analogue.



Let's recall our earlier graph in three dimensions of the curl.



From this plot, we can see that at any given input (x, y), the magnitude of curl does not change no matter how z fluctuates. And, by our analysis, the curl only points in the $z-$axis, straight up along any given (x, y) point. This means that the flux is zero–the divergence is zero–across the entire function. Note that this example provides a good geometric intuition as to the physical meaning behind $\text{div}(\text{curl}\boldsymbol{F}) = 0$). These geometric ideas would generalize (rotate) as you added a z-component to the vector field, leaving the curl with 3 components. We have shown the math works out, and it is possible (albeit difficult) to visualize these shifts in other examples.

To generalize these geometric ideas, the curl measures the movement inward and around–towards the center as any circular path is drawn–while the divergence measures the pull away from it. Going back to our picture of the ball, divergence measures how the ball will be stretched/squished and curl measures its rotation. These concepts and their corresponding mathematical values have no overlaps, being, in a largely figurative sense, orthogonal to one another. Thus, the curl of an arbitrary vector field has no residual measure of divergence left from its original vector field. Or said another way, the flux of the flow of a vector field is zero. However, this geometric link between $\boldsymbol{F}$ and our final divergence of the curl is mathematically difficult to describe with our current tools.

To that end, we have a more direct statement of geometric meaning we can apply in this case. We can say that the vector field defined by the curl of an arbitrary vector field is incompressible across its entire domain. This special class of vector fields defined by the curl of another arbitrary vector field have the property of being incompressible.

6 Conclusion

Altogether, in this challenge problem report, we looked at conservative vector fields and their corresponding potential functions. We saw how the contour plot of potential functions are perpendicular to the gradient (to the vector field) and that said functions differ by a constant. We then took our understanding to divergence and curl, using a variety of examples to relate, divergence, curl, and conservative vector fields (by the gradient). Throughout this, we worked to have a strong geometric interpretation of these relationships. In writing this report, I borrowed ideas from some of my classmates including Aiden, Lucas, and Hero in addition to ideas from office hours. I took ideas for how to prove a convex subset from math stack exchange. Furthermore, I adapted code from an online source to make my vector fields, primarily from this website.