32BH Challenge Problem Report 2

Brendan Connelly

February 20, 2023

Abstract

The following Challenge Problem Report will look at vanishing loci and how they can be used to recognize differentiable k-manifolds embedded in $\mathbb{R}^n$. While the tools we have developed in lecture such as using parameterization allow us to work with manifolds well, some of these tools do not allow us to recognize manifolds very easily. Exploring the concept of vanishing loci will allow us to expand our tool kit when it comes to the powerful tool that are manifolds. We will focus heavily on what we will define as theorem 2.2, particularly on the idea of surjectivity of the Jacobian. Overall, we will use our analysis to build our ability to work with manifolds.

1 Surjectivity and Differentiable Map

In this section, we will look at a differentiable map $F : \mathbb{R}^n \to \mathbb{R}$. A critical hypothesis when considering vanishing loci is that the derivative of the function $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ of which we are taking the vanishing locus of must be surjective onto the manifold, which in the case of F defined in this way is a k-dimensional manifold living in $\mathbb{R}^n$. In this section, we will attempt to show the relationship between the values of the partial derivatives of F. However, first, we will need to consider the following definitions.

Definition 1.1 (Differentiable k-dimensional Manifold Embedded in $\mathbb{R}^n$). *A subset $M \subset \mathbb{R}^n$ is a differentiable k-dimensional manifold embedded in $\mathbb{R}^n$ if for all $\boldsymbol{x} \in M$, there exists an open neighborhood U such that $M \cap U$ is the graph of a C^1 mapping $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$.*

Definition 1.2 (Vanishing Loci). *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. The vanishing locus of f (sometimes called the locus, or the zero locus) is the set of points $V(f)$ where f vanishes. That is,*

$$V(f) = \{\boldsymbol{x} \in A \mid f(\boldsymbol{x}) = 0\}$$

Definition 1.3 (Surjective). *A map $f : X \to Y$ is **surjective** if for every $\boldsymbol{y} \in Y$, there exists an $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) = \boldsymbol{y}$.*

Theorem 1.4 (Equivalent Statements on Surjectivity). *The following statements are equivalent about a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A:*

1. *T is surjective.*
2. *The columns of A span $\mathbb{R}^m$.*
3. *For every $\boldsymbol{b} \in \mathbb{R}^m$, there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \boldsymbol{b}$.*
4. *The rows of A are linearly independent.*

Now that we have the above theorems and definitions, we can return to our differentiable map $F : \mathbb{R}^n \to \mathbb{R}$. We will attempt to show that the following statements are equivalent.

(a) The derivative of $F, [J_F(\boldsymbol{z})]$ is surjective.

(b) At least one of the partial derivatives $\frac{\partial}{\partial x_i}F$ is non-zero.

To show that two statements are equivalent, we must show that 1. (a) implies (b) and that 2. (b) implies (a). Let's tackle both of these proofs using our above definitions.

1.1 Surjective Derivative Implies a Non-zero Partial Derivative

We are assuming that we have a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $[J_F(\boldsymbol{z})]$ is surjective. We thus want to show that one of the partials $\frac{\partial F}{\partial x_i}$ must be non-zero.

Let's first consider what $[J_F(\boldsymbol{z})]$ looks like for a function with n-inputs and a single output.

$$[J_F(\boldsymbol{z})] = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix}$$

It would be difficult to use the direct definition of surjectivity given in definition 1.3 to prove this idea. Let's instead consider theorem 1.4, specifically the biconditional relationship between a linear map being surjective and the columns of the matrix defining it spanning $\mathbb{R}^m$ where m is the number of rows of the matrix, or the dimension of the output of the map.

In our case with a function $F : \mathbb{R}^n \to \mathbb{R}$ using this relationship, $\{\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \cdots, \frac{\partial F}{\partial x_n}\}$ would need to span $\mathbb{R}$. In order span $\mathbb{R}$, we only need one degree of freedom. This property of $\mathbb{R}$ implies that a minimum of one of these partial derivatives must be non-zero. Then, we could use some $\lambda \in \mathbb{R}$ to scale to cover the entirety of $\mathbb{R}$, ie. $\lambda \frac{\partial F}{\partial x_i}$. Thus, we have show that

$$\text{Whenever the derivative of } F, [J_F(\boldsymbol{z})] \text{ is surjective } \implies \text{ at least one of the partial derivatives } \frac{\partial F}{\partial x_i} \text{ is non-zero.}$$

1.2 Non-zero Partial Derivative Implies a Surjective Map

Now let's consider the converse situation. We are assuming that at least one of the partial derivatives of a function $F : \mathbb{R}^n \to \mathbb{R}$ is non-zero. We want to show that $[J_F(\boldsymbol{z})]$ must be surjective onto $\mathbb{R}$.

Because of the nature of the biconditional statement earlier, we can make the exact same argument but in reverse. We know that since there exists a $\frac{\partial F}{\partial x_i} \neq 0$, this one-dimensional vector acts as a basis for $\mathbb{R}$, spanning it. Thus, from our equivalent statements in theorem 1.4, $[J_F(\boldsymbol{z})]$ must be surjective. We have shown that

$$\text{Whenever at least one of the partial derivatives } \frac{\partial F}{\partial x_i} \text{ is non-zero. } \implies \text{ the derivative of } F, [J_F(\boldsymbol{z})] \text{ is surjective}$$

Taken together, we have shown that the following are equivalent statements:

(a) The derivative of $F, [J_F(\boldsymbol{z})]$ is surjective.

(b) At least one of the partial derivatives $\frac{\partial}{\partial x_i} F$ is non-zero.

This is an important building block before we dive into the ideas of using vanishing loci to determine manifolds. We will often define functions $F : \mathbb{R}^n \to \mathbb{R}$ as this could define an $n-1$ dimensional manifold in $\mathbb{R}^n$.

2 Unit Circle as a 1-manifold in $\mathbb{R}^2$

Let's now consider our first example of a manifold. In this section, we will attempt to show that the unit circle $S^1 := \{(x, y) \mid x^2 + y^2 = 1\}$ is a 1-manifold embedded in $\mathbb{R}^2$. As the primary topic in this report, we will use the concept of vanishing loci to do this. We already have two definitions relevant to this defined in section 1 (definition 1.1 and 1.2). However, we will need a couple more to tackle this problem. Consider the following

Theorem 2.1 (Locally showing a vanishing locus is a differentiable manifold). *Let M be a subset of $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open, and let $F : U \to \mathbb{R}^{n-k}$ be a C^1-mapping such that*

$$M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$$

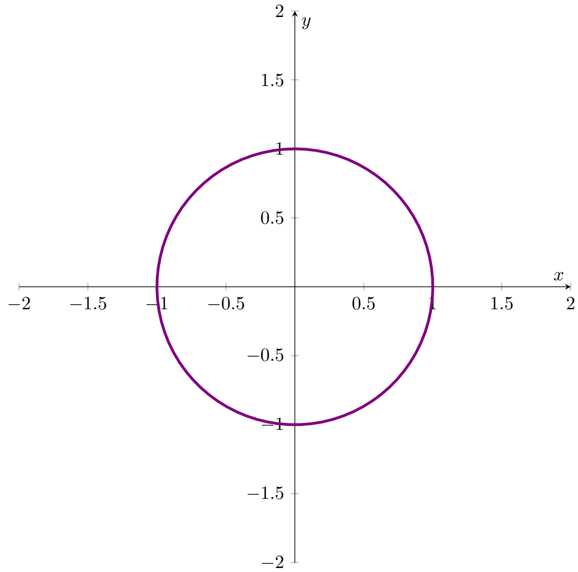
If the derivative $[J_F(\boldsymbol{z})]$ is a surjective map for every $\boldsymbol{z} \in M \cap U$, then $X \cap U$ is a differentiable k-dimensional manifold embedded in $\mathbb{R}^n$.

Theorem 2.2 (Showing a vanishing locus is a differentiable manifold). *Let M be a subset of $\mathbb{R}^n$. If for every $\boldsymbol{z} \in M$, there exists an open set $U \subset \mathbb{R}^n$ containing $\boldsymbol{z}$, and a C^1-mapping $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that*

$$M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$$

and $[J_F(\boldsymbol{z})]$ is surjective for every $\boldsymbol{z} \in M$, then M is a differentiable k-dimensional manifold.

In order to show that S^1 is a differentiable 1-manifold in $\mathbb{R}^2$, we need to closely follow our theorem 2.2 as this will globally show that the unit circle is a manifold. First, recall that we did this in lecture using definition 1.1, splitting up this circle into four parts. This should be a more effective way of determining that S^1 is a manifold.



2.1 Proof Introduction

First, we should choose $M \subset \mathbb{R}^2$ as this matches the dimension we want our subset to be embedded in. We should–and have the ability to–choose

$$M = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$$

Now, we need to show that for every $\boldsymbol{z} = (x, y) \in M$, there exists an open set $U \subset \mathbb{R}^2$ containing $\boldsymbol{z}$ and a C^1 mapping $F : U \to \mathbb{R}$ such that

$$M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \mathbf{0}\}$$

where $[J_F(x, y)]$ is surjective for every $(x, y) \in M$, then M is a differentiable 1-dimensional manifold.

Now, let's show that this holds true for our choices of U and F.

2.2 Finding a U and F

First, let's choose $U = \mathbb{R}^2$. We can adjust this as needed if we encounter any problems.

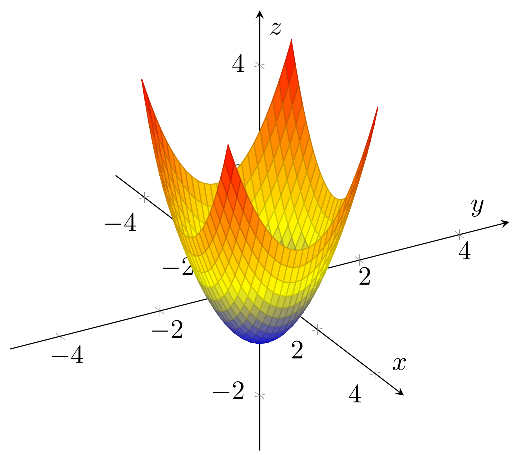
Now, let's consider our function F. We want F to input x and y and output a real number. The vanishing locus of this function should intersect with U to make M. Since we defined $U = \mathbb{R}^2$, we will have

$$S^1 = M = M \cap U \ = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \mathbf{0}\}$$

With this in mind, let's define $F(x, y)$ based on a modification of our definition of S^1. Let

$$F(x, y) = x^2 + y^2 - 1$$

It should be clear that this is a C^1 mapping as the derivatives of this function defined by two polynomials are by definition differentiable and their derivatives continuous. Here is $\Gamma_{F(x,y)} \subseteq \mathbb{R}^3$ to provide visual context for our analysis.



2.3 Checking Surjectivity

Now that we have made our two choices, we see that $M \cap U$ is indeed the vanishing locus of F. Now, if we can show that $[J_F(\boldsymbol{z})]$ is surjective for every $(x, y) \in M$, then M is a differentiable 1-dimensional manifold.

Consider

$$[J_F] = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

This connects directly to our work in the previous section. We know that at least one of the $\frac{\partial F}{\partial x_j} \neq 0$ if $[J_F(\boldsymbol{z})]$ is surjective. From our calculation of $[J_F(\boldsymbol{z})]$, we know that this will only not be the case at $(x, y) = (0, 0))$. This point is not on the manifold and is thus not a problem. As such, $[J_F(\boldsymbol{z})]$ is surjective for every $(x, y) \in M$. Therefore, M is a differentiable 1-manifold embedded in $\mathbb{R}^2$. □

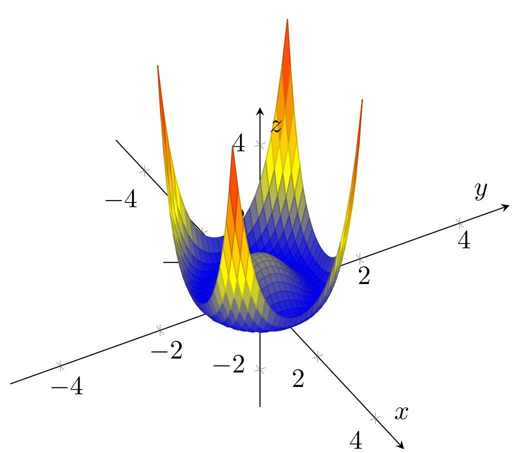
This section has offered insights into using a vanishing locus to determine whether a function describes a manifold. It shows the importance of surjectivity of the Jacobian as this allows the Jacobian to describe all of the manifold, which is a critical feature. As we progress, we will attempt to generalize our understanding of recognizing manifolds through vanishing loci.

3 Example Challenging Notion of Need for Surjectivity

While we have put a lot emphasis into the importance of F being surjective when defining manifolds via vanishing loci, we need to make sure we take a nuanced approach to abiding by our definition. The section's conclusions will come down to the fact that F is not unique in defining manifolds through vanishing loci.

For this section, we will not need to consider any additional definitions. Consider the following function

$$F(x, y) = (x^2 + y^2 - 1)^2$$



3.1 Showing the vanishing locus of F is a manifold

From definition 1.2, we can see that the vanishing locus of F is

$$\{(x, y)|(x^2 + y^2 - 1)^2 = 0\}$$

By simply taking the square root of both sides, we can see that this set reduces to

$$\{(x, y) \mid x^2 + y^2 = 1\}$$

We know that this is equivalent to S^1. In fact, it is how S^1 is defined. We have already shown that this is a manifold in section 2. Thus, the vanishing locus of F is a manifold.

3.2 Showing F is not Surjective

If we were to assume that F was unique in theorem 2.2, $[J_F(\boldsymbol{z})]$ would need to be surjective onto $\mathbb{R}$. However, we will show that this is not the case. Let's calculate $[J_F(\boldsymbol{z})]$. We can take the partial derivatives of $F(x, y) = (x^2 + y^2 - 1)^2$ without expanding out our expression, which will give us unique insight into its values on M. Suppose $\boldsymbol{z} \in M$

$$\frac{\partial F}{\partial x} = 4x(x^2 + y^2 - 1)$$
$$\text{On } M = \{(x, y) \mid x^2 + y^2 - 1 = 0\}$$
$$\frac{\partial F}{\partial x} = 4x(0)$$
$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = 4y(x^2 + y^2 - 1)$$
$$\text{On } M = \{(x, y) \mid x^2 + y^2 - 1 = 0\}$$
$$\frac{\partial F}{\partial y} = 4y(0)$$
$$\frac{\partial F}{\partial y} = 0$$

$$[J_F] = \begin{bmatrix} 4x(x^2 + y^2 - 1) & 4x(x^2 + y^2 - 1) \end{bmatrix}$$

$$[J_F(\boldsymbol{z})] = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

We can see from our work in section 1 that $[J_F(\boldsymbol{z})]$ is not surjective on $\mathbb{R}$ as it is always $\mathbf{0}$ for all $\boldsymbol{z} \in M$.

Let's consider how this is possible. We have the same manifold embedded also in $\mathbb{R}^2$ as we did in section 2. It is very clearly defined as S^1. This is possible because F is not unique. Multiple different functions can have the same vanishing locus. The wording of theorem 2.2 is that there exists a C^1 mapping F, not that said F is unique or works for any F that has the same vanishing locus. This is an important nuance we should remember going forward and a critical exposition of the theorem.

4 Recognizing Manifolds from a Collection of Functions

In this section we will consider the following relation where $c \in \mathbb{R}$.

$$x^3 - 3x + y^3 - 3y = c$$

4.1 Finding c values

We want to again consider theorem 2.2 in order to determine when our relation describes a manifold through the idea of a vanishing locus. This manifold will be a 1-manifold embedded in $\mathbb{R}^2$, just as S^1 is.

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^3 - 3x + y^3 - 3y - c = 0\}$$

Again, similar to what we did in section 2, let's make a guess at a potential $U \subset \mathbb{R}^2$ and a C^1 mapping $F : \mathbb{R}^2 \to \mathbb{R}$. We should be able to choose $U = \mathbb{R}^2$ again and the map $F(x, y) = x^3 - 3x + y^3 - 3y - c$ to serve our purpose. F is differentiable, and its derivative is continuous. Observe that

$$M = M \cap U = \{(x, y) \in \mathbb{R}^2 \mid x^3 - 3x + y^3 - 3y - c = 0\}$$

Thus, all we have to do is consider the surjectivity of the Jacobian of F. Our analysis of this will give us the values of $c \in \mathbb{R}$ that are 1-manifolds in $\mathbb{R}^2$ and c values that potentially may not produces a manifold. When we find those c values, we will have to do a little work to see why our subset is not a manifold.

Consider $[J_F]$

$$[J_F] = \begin{bmatrix} 3x^2 - 3 & 3y^2 - 3 \end{bmatrix}$$

Recall from our work in section 1, we simply want $[J_F(\boldsymbol{z})]$ to surject onto $\mathbb{R}$ where $\boldsymbol{z} \in M$, meaning one of these has to be non-zero. Thus, from setting each partial derivative equal to 0, it should be clear that our condition of surjectivity is not met at the following four points:

$$(x, y) = (-1, -1), (-1, 1), (1, -1), (1, 1)$$

Let's find the c values such that the vanishing locus of $F(x, y) = x^3 - 3x + y^3 - 3y - c$ contains these points. Plugging in we see that we have

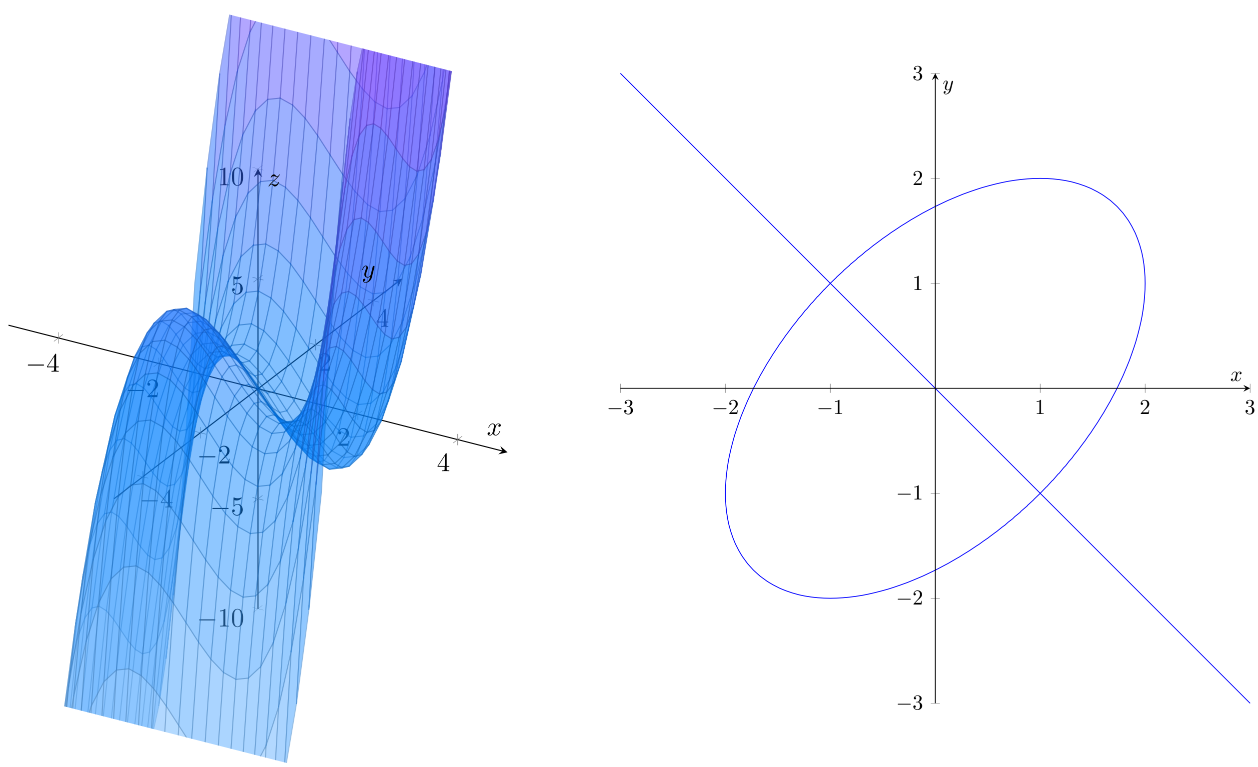
1. $(x, y) = (1, 1)$ produces $1 - 3 + 1 - 3 = -4 \implies c = -4$
2. $(x, y) = (1, -1)$ produces $1 - 3 - 1 + 3 = 0 \implies c = 0$
3. $(x, y) = (1, -1)$ produces $-1 + 3 + 1 - 3 = 0 \implies c = 0$
4. $(x, y) = (-1, -1)$ produces $-1 + 3 - 1 + 3 = 4 \implies c = 4$

We have three distinct values of c for which $[J_F(\boldsymbol{z})]$ is not surjective. We have thus shown that for all $c \in \mathbb{R} \setminus \{-4, 0, 4\}$, M is a manifold.

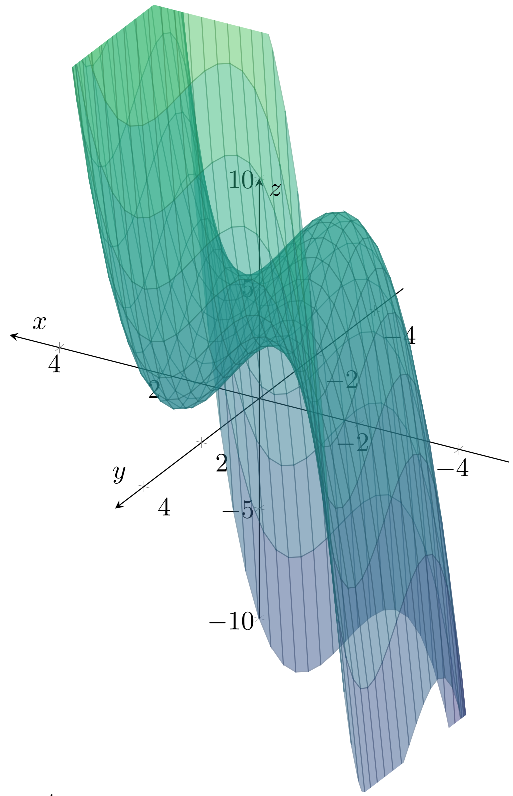
4.2 Showing the Vanishing Loci are not Manifolds for Specific c values

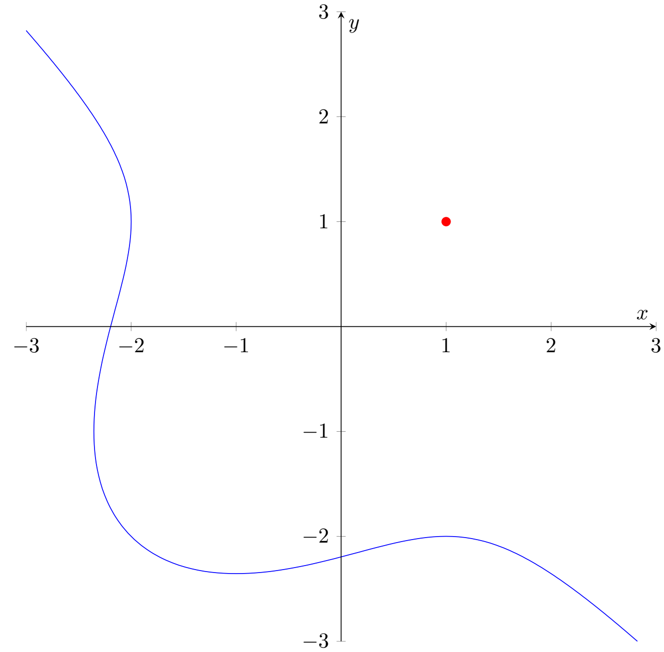
Now, let's consider our three values of c and show that their corresponding vanishing loci indeed do not describe manifolds. We can essentially prove this by graphing each vanishing loci in one go. We will thus do that as a partial proof and some context followed by a qualitative explanation of what it means to be a 1-manifold in $\mathbb{R}^2$. We will look at all three values of c together. First consider the following plots of $\Gamma_F(x, y)$ on the left and the vanishing locus of $F(x, y)$ on the right.

1. $c = 0$

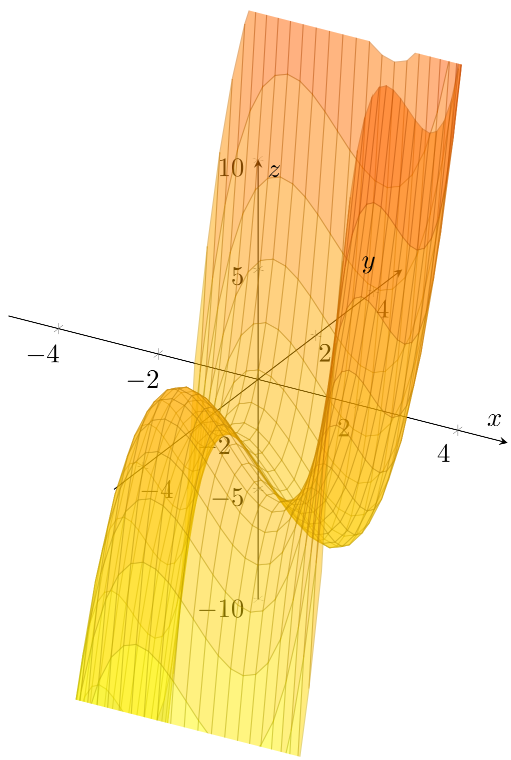


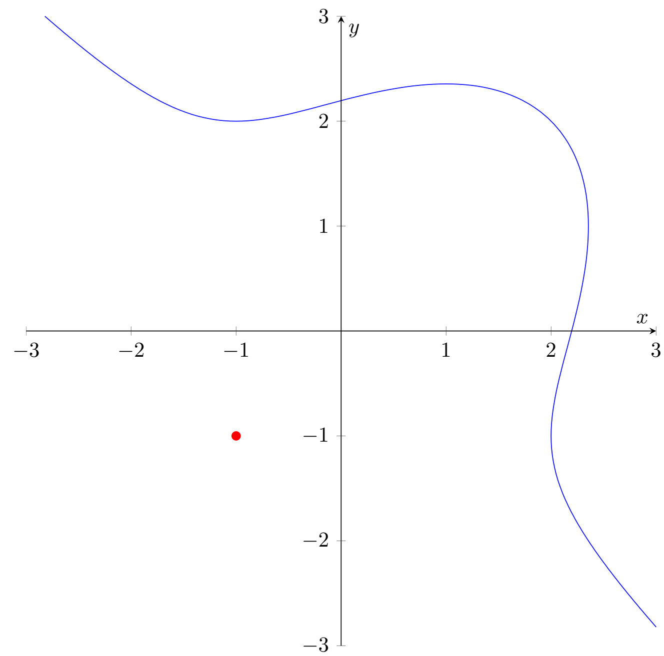
2. $c = -4$





3. $c = 4$





Now that we have the above graphs for reference, let's go through the $c = 0$ case and then analyze $c = -4, 4$ case together.

We learned in lecture that curves cannot intersect if they are to be manifolds. This is clearly the case for $c = 0$. Going all the way back to definition 1.1, we know that at $(-1, 1)$ and $(1, -1)$ the graph does not look like the graph of a function. It is impossible for a function to be going into those two branches. So, as close as you go to those regions, you will never be able to satisfy definition 1.1.

Now, looking at $c = -4, 4$, we see that both have a single point off of the curve that satisfies the relation. From the 3 dimensional graph, we can see that a single point intersects the xy-plane in one neighborhood. This point cannot locally look the graph of a function from $\mathbb{R} \to \mathbb{R}$ as prescribed by definition 1.1. A single point cannot behave in this way, no matter how close you zoom in. Furthermore, we know that a single point is actually a 0-manifold. We know we can combine manifolds of the same dimension that don't intersect and call it a singular manifold. However, a single point and a curve do not have the same dimension, and thus our vanishing loci are not manifolds.

Taken together, we can conclude that

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^3 - 3x + y^3 - 3y - c = 0\}$$

describes a manifold for all $c \in \mathbb{R}$ except for when $c = -4, 0, 4$.

This analysis is critical in developing our understanding of manifolds. We have gained a greater appreciation for the nuance of the condition of surjectivity and its importance in recognizing manifolds. We were able to prove that the vast majority of cases of c were manifolds merely by demonstrating this condition. With a little more work, it was easy to see that the remaining cases describes subsets that had irreconcilable problems, preventing them from being manifolds.

5 Connecting Manifolds to Critical Points

In this section, we will build on our work from the previous section and connect our findings of c to our understanding of critical points in differential multivariable calculus. Recall the following definitions:

Definition 5.1 (Critical Points). *A point $P \in \mathbb{R}^n$ is a critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ if either*

(a) $Df(\boldsymbol{P}) = 0$

(b) $Df(\boldsymbol{P})$ *does not exist*

Definition 5.2 (Hessian Matrix). *The Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}$ at $\boldsymbol{x_0}$ is*

$$[H_f(\boldsymbol{x_0})] = \begin{bmatrix} D_1D_1f(\boldsymbol{x_0}) & D_2D_1f(\boldsymbol{x_0}) & \cdots & D_nD_1f(\boldsymbol{x_0}) \\ D_1D_2f(\boldsymbol{x_0}) & D_2D_2f(\boldsymbol{x_0}) & \cdots & D_nD_2f(\boldsymbol{x_0}) \\ \vdots & \vdots & \ddots & \vdots \\ D_1D_nf(\boldsymbol{x_0}) & D_2D_nf(\boldsymbol{x_0}) & \cdots & D_nD_nf(\boldsymbol{x_0}) \end{bmatrix},$$

where $D_iD_jf(\boldsymbol{x_0})$ denotes the second partial derivative of f with respect to x_i and then x_j at $\boldsymbol{x_0}$.

Theorem 5.3 (Second Derivative Test). *Let $\boldsymbol{x_0} \subset U$ be a critical point of $f(x, y) : U \to \mathbb{R}$, and suppose that f is in $C^2(U)$. Let us write $D = \det[H_f(\boldsymbol{x_0})]$.*

a. *If $D > 0$ and $f_{xx}(\boldsymbol{x_0}) > 0$, then there is a local minimum at $\boldsymbol{x_0}$.*

b. *If $D > 0$ and $f_{xx}(\boldsymbol{x_0}) < 0$, then there is a local maximum at $\boldsymbol{x_0}$.*

c. *If $D < 0$, then f has a saddle point at $\boldsymbol{x_0}$.*

d. *If $D = 0$ or does not exist, then the test is inconclusive.*

Let's use these tools to analyze the critical points of $f(x, y) = x^3 - 3x + y^3 - 3y$. Let's calculate both the Jacobian, the critical points, and the Hessian.

$$[J_f] = \begin{bmatrix} 3x^2 - 3 & 3y^2 - 3 \end{bmatrix}$$

Thus, the critical points are the following four points $(x, y) = (\pm 1, \pm 1)$. This should be very familiar.

$$[H_f] = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

Note that the $\det([H_f]) = 36xy$

Observe that when $(x, y) = (-1, 1)$ or $(1, -1)$, we have a negative determinant and thus a saddle point. This fits with our graph. When $(x, y) = (-1, -1)$ or $(1, 1)$, we have a positive determinant and thus a local maxima or minima which also fits with our first 3 dimensional figure in section four.

There is a connection between these critical points and our choices of c. Specifically, we will start with our choices $c = -4, 4$, which did not satisfy the necessary condition of surjectivity. We see that when we evaluate our local maximum and minimum

$$f(-1, -1) = 4 \text{ and } f(1, 1) = -4$$

This is the same math we did early when checking the surjectivity condition. The local maximum and minimum take the same value as our c values that failed our test of surjectivity. This is a direct connection between our work in section four and our analysis of critical points.

Furthermore, when we consider $(x, y) = (-1, 1)$ or $(1, -1)$, we get saddle points of

$$f(-1, 1) = 0 \text{ and } f(1, -1) = 0$$

This matches our other value of c that failed the necessary conditions, $c = 0$. It is even worth mentioning that we found two saddle points with this value, the same way we found two points that produces $c = 0$ in section four–we have a similar duplicate result.

This shows that we are essentially doing the same math to demonstrate surjectivity as we are to finding critical points. These critical points fail surjectivity either by pinching (which should be vaguely reminiscent of finding saddle points on a contour plot) or by intersecting the xy-plane at a single point. The latter of which we get by simply shifting $f(x, y)$ vertically along the z-axis. As we generalize in the following section, the existence of these critical points don't guarantee that a manifold doesn't exist at the corresponding c-values. For instance, a maximum could create a 0-manifold in $\mathbb{R}^2$. However, it can be an important indicator of where to look.

All in all, in this section, we have made a great deal of progress in connecting our understanding of when $F(x, y)$ in section described a manifold to the value of its local maximum, minimum, and saddle points through combining our understanding of differential and integral multivariable calculus. This is a useful connection to wrap up our study of the use of manifolds, providing a useful tool to help recognize these powerful mathematical objects.

6 Food for Thought

In this section, we will briefly generalize some of the exercises we have done as of yet, providing a very brief explanation.

6.1 n-sphere is a Manifold

We will attempt to show that the n-sphere (defined by $\sum_{i=1}^{n} x_i^2 = 1$) is a differentiable $n - 1$ manifold. This will quickly generalize our work in section 2.

Suppose $\boldsymbol{z} \in \mathbb{R}^n$. Let's choose

$$M = S^{n-1} = \left\{ \boldsymbol{z} = \{x_1, x_2, \cdots, x_n\} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

Guess that $U = \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(\boldsymbol{z}) = -1 + \sum_{i=1}^{n} x_i^2$$

We know that F is a C^1 mapping as a function of polynomials. We now just need to show that the $[J_F(\boldsymbol{z})]$ is surjective for every $\boldsymbol{z} \in M$.

$$[J_F] = \begin{bmatrix} 2x_1 & 2x_2 & \cdots & 2x_n \end{bmatrix}$$

From all the way back in section one, we see that all of these have to be zero for our condition not to be fulfilled. As $\mathbf{0} \notin M$, $[J_F(\boldsymbol{z})]$ is surjective for every $\boldsymbol{z} \in M$. Thus, we have shown that n-sphere (defined by $\sum_{i=1}^{n} x_i^2 = 1$ and is S^{n-1} by how we have defined it previously) is a differentiable $n - 1$ manifold.

6.2 Generalization of section three

Suppose that there exists a C^1-mapping $F : U \to \mathbb{R}^{n-k}$ such that $F(\boldsymbol{z}) = \mathbf{0}$ defines a manifold.

(a) Show that $(F(\boldsymbol{z}))^2 = \mathbf{0}$ defines the same manifold.

(b) Show that $[J_F(\boldsymbol{z})]$ is never surjective.

(a) $(F(\boldsymbol{z}))^2 = \mathbf{0}$ defines the same manifold.

We have been defining manifolds through the idea of vanishing loci. The vanishing locus of $(F(\boldsymbol{z}))^2$ is the same as that of $F(\boldsymbol{z})$. We can see this by taking the square root of both sides. We are left with the exact same manifold. Because we supposed the vanishing locus of $F(\boldsymbol{z})$ to be a manifold, $(F(\boldsymbol{z}))^2$ is as well.

(b) $[J_F(\boldsymbol{z})]$ is never surjective

We can calculate the partial derivatives using the chain rule.

$$[J_F(\boldsymbol{z})] = \begin{bmatrix} 4x_1\,(F(x,y)) & 4x_2\,(F(x,y)) & \cdots & 4x_n\,(F(x,y)) \end{bmatrix}$$

Recall that $F(x, y) = 0$ is the region we are interested in, meaning we have a derivative with all zeros, which is thus not surjective. Yet, this still defines a manifold and is a nuance we discussed earlier.

6.3 Critical Points

We want to answer the following. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^1$ be a C^1-mapping. What is the relationship between a locus $F(\boldsymbol{z}) = c$ and the critical points of F?

From our work in section five, we saw that the critical points of F occur at points that cause $[J_F(\boldsymbol{z})]$ not to be surjective. If these points are on the locus $F(\boldsymbol{z}) = c$, then that locus may not be a manifold, potentially either pinching or containing a stray point. At saddle points, we claim that the vanishing loci will never be a manifold. Yet, at minimums and maximums, such as singular points and the scenario in section 3, the vanishing loci may be a manifold nonetheless.

7 Conclusion

In this challenge problem report, we analyzed what it takes for a map to be surjective onto $\mathbb{R}$ and the implications of that. Then, we considered the unit cirlce as a 1-manifold in $\mathbb{R}^2$, followed up with a challenge to surjectivity and an added nuance in section three. Lastly, in sections four and five, we considered a family of functions that could be shifted along the z-axis and looked at when its vanishing locus was a manifold. Overall, we looked at useful examples that allowed our understanding of manifolds to grow. Along the way, I took a few ideas from office hours and from our class GroupMe, but otherwise I did most of the math myself. I took theorems and definitions from the Challenge Problem Report, Lecture Notes, and my own notes.