32BH Challenge Problem Report 1

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Abstract

This Challenge Problem Report will explore improper integrals, meaning integrals of functions that are unbounded or do not have bounded support. We have discovered how we can set up integrals as a limit of Upper and Lower Darboux sums, squeezing the integral as N approaches infinity. However, in doing this, we needed our function to be bounded and have bounded support. Yet, it is possible to take integrals beyond these limitations. We will thus explore the nuances of pushing aside these limitations.

1 Introduction to Improper Integral of an Unbounded Function

Through the use of Upper and Lower Darboux Sums, we were able to rigorously define an integral which is bounded with bounded support. Furthermore, we know how to adjust bounds of integration through the use of the indicator function and take the integral on set regions. In this section, we will explore an example of a function that is not bounded and does not have bounded support.

Below are the definitions of bounded and bounded support for a function.

Definition 1.1: *A subset $D \subset \mathbb{R}^n$ is bounded if there exists some $r > 0$ such that $D \subset B_r(\mathbf{0})$.*

Definition 1.2:*A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is bounded if its image $\{f(\boldsymbol{x}) \,|\, \boldsymbol{x} \in A\}$ is a bounded subset of $\mathbb{R}$.*

Definition 1.3: *A function has bounded support if its support is bounded where*

$$\text{supp}(f) := \overline{\{\boldsymbol{x} \in A \,|\, f(\boldsymbol{x}) \neq 0\}}$$

and the closure of some subset D is defined as

$$\overline{D} = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, B_r(\boldsymbol{x}) \cap D \neq \emptyset \text{ for all } r \in \mathbb{R}\}$$

Equivalently, a function is defined as having bounded support if there exists $R > 0$ such that $f(\boldsymbol{x}) = 0$ for all $||\boldsymbol{x}|| > R$.

Consider the following function $f : \mathbb{R} \to \mathbb{R}$ where p is a positive real number.

$$f(x) = \frac{1}{x^p}$$

We are going to look at the following integral. Our goal will be to determine the positive real values of p that make this integral converge.

$$\int_0^1 f(x)\; dx$$

Because we have defined the bounds of this integral for a finite region, we do not need to worry about bounded support. This would be adjusted by an indicator function setting everything outside the bounds to 0, leaving the function with bounded support. Thus, we will now consider this function and its integral just on the region $x \in (0, 1)$.

First, lets calculate the indefinite integral using our knowledge from single variable calculus. We will use the power rule and split into two cases.

Case I: $p \neq 1$

$$\int f(x)\; dx = \int \frac{1}{x^p}\; dx = \frac{x^{-p+1}}{-p+1} + C$$

Case II: $p = 1$

$$\int f(x)\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

From a quick glance at $f(x)$, we can easily deduce that at $x = 0$, the function is undefined and thus would be unbounded here. To tackle this, we will need the following definitions.

Definition 1.4: *A function $f : \mathbb{R} \to \mathbb{R}$ is* *<u>continuous</u>* *at $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$.*

Definition 1.5: *Let $f(x)$ be a function that is continuous but unbounded on the open interval (a, b). Let $\alpha, \beta \in (a, b)$ such that $a < \alpha < \beta < b$. Then we can compute the integral*

$$I(\alpha, \beta) := \int_{\alpha}^{\beta} f(x)\,dx$$

If the limit $\lim_{(\alpha,\beta)\to(a,b)} I(\alpha, \beta) = L$ exists, then we can define the improper integral $\int_a^b f(x)\,dx$ to be L. That is,

$$\int_a^b f(x)\,dx := \lim_{\substack{\alpha\to a\\ \beta\to b}} \int_{\alpha}^{\beta} f(x)\,dx$$

As we are dealing with an unbounded integral, we will use definition 1.5 to analyze our problem. From our definition, we want to find when the following limit to exists.

$$\lim_{\substack{\alpha\to 0\\ \beta\to 1}} \int_{\alpha}^{\beta} \frac{1}{x^p}\,dx = \lim_{\alpha\to 0} \int_{\alpha}^{1} \frac{1}{x^p}\,dx$$

Let's use the indefinite integral we found for **Case I** and attempt to evaluate our limit.

$$\begin{aligned}\lim_{\alpha\to 0} \int_{\alpha}^{1} \frac{1}{x^p}\,dx &= \lim_{\alpha\to 0} \left.\frac{x^{-p+1}}{-p+1}\right|_{\alpha}^{1} \\ &= \frac{1}{-p+1} - \lim_{\alpha\to 0} \frac{\alpha^{-p+1}}{-p+1}\end{aligned}$$

Since we are working with a finite p, we will need $\lim_{\alpha\to 0} \alpha^{-p+1}$ to converge to 0 for this limit to exist. The only way for this to happen is α is not raised to a negative power. Thus, we want

$$-p + 1 > 0$$

Because we defined p to be positive, with our current case, we have found that

$$0 < p < 1$$

are the values on which $\int_0^1 f(x)\,dx$ converges.

From our integral in **Case II**, we have

$$\lim_{\alpha\to 0} \left(\ln|x|\right)\Big|_{\alpha}^{1}$$

We know from single variable calculus that $\lim_{\alpha\to 0} \ln|\alpha|$ diverges to negative infinity. Thus, this integral does not exist.

From our work, we can see that p, defined as being a positive real number, must satisfy $p \in (0, 1)$ in order for our integral to exist. Through our variable p, we have seen how $f(x)$ can have a finite definite integral when it is unbounded in some cases but not in other cases. This hints at the need for a nuanced approach to unbounded integrals which we will look at in the following sections.

2 Integrals with Unbounded Support

In the last section, we looked at integrals that were unbounded but had bounded support in the domain in which we were integrating. In most of the sections in this report, we will look at integrals that do not have bounded support. In order to do this, we will look at the same integral twice, with different limits of integration.

We will need the following definition.

Definition 2.1: *Let $a \in \mathbb{R}$, and suppose that $f(x)$ is bounded and integrable on $[a, b]$ for every $b > a$. Then we can compute the integral*

$$\int_a^b f(x)\ dx$$

If the limit $\lim_{b\to\infty} \int_a^b f(x)\ dx = L$ exists, then we can define the improper integral $\int_a^\infty f(x)\ dx$ to be L. That is,

$$\int_a^\infty f(x)\ dx := \lim_{b\to\infty} \int_a^b f(x)\ dx$$

In this section, we will consider the following integral:

$$\int \frac{1}{x^2+1}\ dx$$

From single variable calculus, we know

$$\int \frac{1}{x^2+1}\ dx = \arctan(x) + C$$

i.A Finite Limit of Integration

We will first consider these bounds of integration and attempt to evaluate it in terms of B.

$$\int_0^B \frac{1}{x^2+1}\ dx$$

We can do the following

$$\begin{aligned}\int_0^B \frac{1}{x^2+1}\ dx &= [\arctan(x)]|_0^B \\ &= \arctan(B) - \arctan(0) \\ &= \arctan(B)\end{aligned}$$

Because $y = \arctan(x)$ is continuous across $\mathbb{R}$, we know that a finite B will produce a finite value for the integral. Thus, we have shown that

$$\int_0^B \frac{1}{x^2+1}\ dx = \arctan(B)$$

ii. Improper Integral

Our logical next step in our analysis of functions with unbounded support is to consider the integral from 0 to ∞.

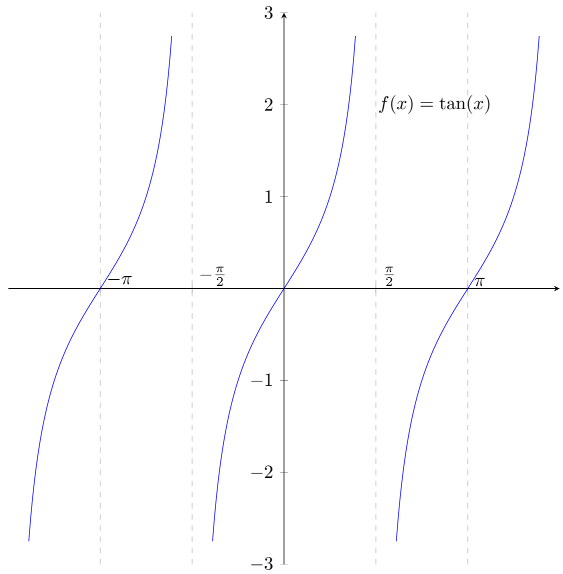
Thus, consider

$$\int_0^\infty \frac{1}{x^2+1}\ dx$$

From our work in part i and definition 2.1, this should produce

$$\lim_{\beta\to\infty} \arctan(\beta)$$

From single variable calculus, we know that this limit has a value of $\frac{\pi}{2}$. If it helps, we can look at the graph of $y = \tan(x)$ and see where it has vertical asymptotes.



We can see graphically that at $\frac{\pi}{2}, \tan(x)$ shoots off to infinity. We can see that, being a trigonometric function, $\tan(x)$ is periodic, but given how $\arctan(x)$ is defined (which in itself gives the answer to this limit), the answer we are looking for is $\frac{\pi}{2}$

$$\lim_{\beta\to\infty} \arctan(\beta) = \frac{\pi}{2}$$

We have now looked at an example of a function without bounded support in addition to our earlier example of an unbounded function. Again, we see that it is possible to integrate a function that does not satisfy a condition that we derived in class–bounded or bounded support–but doing so can only happen in specific cases and requires nuance.

3 Generalization of Definition 2.1

In this section, we will attempt to generalize the one way, one limit definition that is definition 2.1 to tackle integrals in the form

$$\int_{-\infty}^{b} f(x)\ dx$$

where $f(x)$ is bounded and integrable on $[a, b]$ for every $a < b$ and where $b \in \mathbb{R}$. This will further develop our ability to deal with functions that do not have bounded support.

We will first make a claim–which we will call definition 3.1–and then attempt to explain why it is true.

Definition 3.1: *Let $b \in \mathbb{R}$, and suppose that $f(x)$ is bounded and integrable on $[a, b]$ for every $b > a$. Then we can compute the integral*

$$\int_{a}^{b} f(x)\ dx$$

If the limit $\lim_{a\to-\infty} \int_a^b f(x)\ dx = L$ exists, then we can define the improper integral $\int_{-\infty}^{b} f(x)\ dx$ to be L. That is,

$$\int_{-\infty}^{b} f(x)\ dx := \lim_{a\to-\infty} \int_{a}^{b} f(x)\ dx$$

If definition 2.1 is true, we should very easily be able to generalize it to the above definition. Symmetry should intuitively exist in approaching infinity in either the negative or positive end of the x-axis. To confirm, lets define a function $h(x) = f(-x)$. In this function, our original definition for $f(x)$ states

$$\int_{a}^{\infty} f(x)\ dx := \lim_{b\to\infty} \int_{a}^{b} f(x)\ dx$$

If we put in $h(x)$,

$$\int_{-a}^{-\infty} h(x)\ dx := \lim_{b\to-\infty} \int_{-a}^{b} h(x)\ dx$$

By known properties of integrals from single variable calculus, we have

$$\Rightarrow -\lim_{b\to-\infty} \int_{b}^{-a} h(x)\ dx$$

From here, we can substitute our original variables by redefining out functions and constants, returning to

$$\int_{-\infty}^{b} f(x)\ dx := \lim_{a\to-\infty} \int_{a}^{b} f(x)\ dx$$

This is by no means a rigorous proof, but should provide an intuition for this definition. In general, the symmetry of this definition with 2.1 should be sufficient proof. The above work is an analogue to graphing a function and showing that the integral is the same if the function is flipped. We have thus generalized our definition and can begin to analyze what it would mean to then combine definition 2.1 and 3.1.

4 Cauchy Principal Value and Agreement with Improper Integral

In this section, we will introduce the idea of the Cauchy principal value. This will look at the nuances of the combination of definitions 2.1 and 3.1. This will again further our work analyzing integrals of functions without bounded support. First lets look at two definitions, one of the improper integral with infinity and negative infinity as limits of integration and then that of the Cauchy principal value. After that, we will begin our work analyzing their respective meanings and equivalences.

Definition 4.1: *Let $f:\mathbb{R}\to\mathbb{R}$ be a function that is continuous on $\mathbb{R}$. Let $\alpha,\beta\in\mathbb{R}$ such that $\alpha<\beta$. Then we can compute the integral*

$$I(\alpha,\beta) := \int_{\alpha}^{\beta} f(x)\ dx$$

If the limit $\lim_{\substack{\beta\to\infty\\ \alpha\to-\infty}} I(\alpha,\beta) = L$ exists, then we can define the improper integral $\int_{-\infty}^{\infty} f(x)\ dx$ as the limit. That is,

$$\int_{-\infty}^{\infty} f(x)\ dx := \lim_{\substack{\beta\to\infty\\ \alpha\to-\infty}} \int_{\alpha}^{\beta} f(x)\ dx = L$$

While this may look similar to definition 1.5, note that it is really a combination of 2.1 and 3.1 because they both address functions without bounded support. Most importantly, the two limits must exist independent of each other and the rates at which both approach infinity.

Definition 4.2: *Let $f:\mathbb{R}\to\mathbb{R}$ be a function that is continuous on $\mathbb{R}$. Then we can compute the integral*

$$I(R) := \int_{-R}^{R} f(x)\ dx$$

If the limit $\lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx = C$ exists, then we can define the ***Cauchy principal value*** *to be C. That is,*

$$P.V. \int_{-\infty}^{\infty} f(x)\ dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx$$

Now that we have established definitions, let's look at a specific integral and calculate both the improper integral and the Cauchy principal value. Consider

$$f(x) = xe^{-x^2}$$

For use later, we can evaluate the indefinite integral of $f(x)$ using our knowledge of single variable calculus.

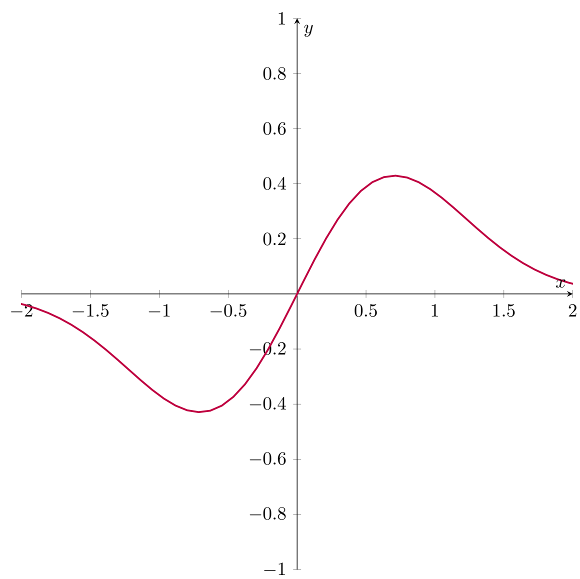
$$\int xe^{-x^2}\ dx = -\frac{1}{2}e^{-x^2} + C$$

i. Cauchy Principal Value

Our first step will be to find the Cauchy principal value of $f(x)$. Looking at our definition, we have

$$\begin{aligned}
P.V. \int_{-\infty}^{\infty} f(x)\ dx &:= \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx \\
&= \lim_{R\to\infty} \int_{-R}^{R} xe^{-x^2}\ dx \\
&= \lim_{R\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]\Bigg|_{-R}^{R} \\
&= \lim_{R\to\infty} \left[-\frac{1}{2}e^{-R^2} - \left(-\frac{1}{2}\right)e^{-R^2}\right] \\
&= \lim_{R\to\infty} \left[-\frac{1}{2}e^{-R^2} + \frac{1}{2}e^{-R^2}\right] \\
&= \lim_{R\to\infty} (0) \\
&= 0
\end{aligned}$$

We have thus found that our Cauchy principal value is 0. This should be relatively clear to see from the algebra, but we can also look graphically. Looking at the graph of $f(x)$,



we can see that it is an odd function and approaching $x = \infty$ and $x = -\infty$ at the same rates will allow them to mutally cancel each other and produce this 0 result.

ii. Improper Integral

To compare with the Cauchy principal value, we will calculate the improper integral. Let's use definition 4.1.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx &:= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \int_{\alpha}^{\beta} f(x)\ dx \\
&= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \int_{\alpha}^{\beta} xe^{-x^2}\ dx \\
&= -\frac{1}{2} \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \left[e^{-x^2}\right]\Big|_{\alpha}^{\beta} \\
&= -\frac{1}{2} \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \left[e^{-\beta^2} - e^{-\alpha^2}\right] \\
&= -\frac{1}{2}[1-1] \\
&= 0
\end{aligned}
$$

We have now calculated both the improper integral and the Cauchy principal value for $f(x)$. In this case, we can see that they agree. As we will explore later, it is worth noting that this occurs because both sides of the limit converged on their own. Furthermore, by definition, if the indefinite integral exists, the Cauchy principle value will agree. We shall call this definition 4.3. Naturally, we will next look at a function when the improper integral does not exist but the Cauchy principal value exists.

5 Cauchy Principal Value and Disagreement with Improper Integral

In this section, we will further expand our understanding of functions without bounded support. In this function

$$g(x) = \frac{2x}{x^2+1}$$

the Cauchy principal value will exist and the improper integral will not. We will see why this is both algebraically and graphically. To recap, the improper integral must exist regardless of the rate at which the limit on either side is reached.

For reference, from single variable calculus, we know that

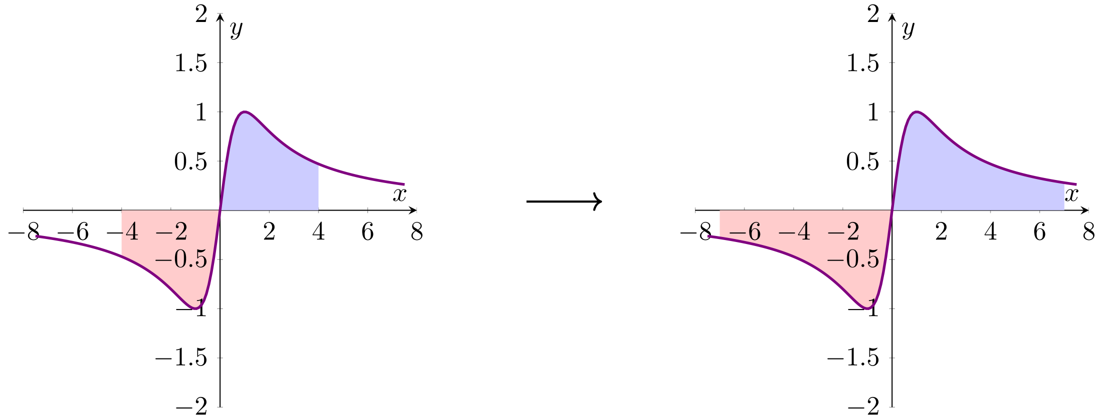
$$\int g(x)\ dx = \int \frac{2x}{x^2+1}\ dx = \ln\left(x^2+1\right) + C$$

i. Cauchy Principal Value

Using definition 4.2 again produces

$$
\begin{aligned}
P.V. \int_{-\infty}^{\infty} g(x)\ dx &:= \lim_{R\to\infty} \int_{-R}^{R} g(x)\ dx \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{2x}{x^2+1}\ dx \\
&= \lim_{R\to\infty} \left[\ln\left(x^2+1\right)\right]\Big|_{-R}^{R} \\
&= \lim_{R\to\infty} \left[\ln\left(R^2+1\right) - \ln\left((-R)^2+1\right)\right] \\
&= \lim_{R\to\infty} [\ln(1)] \\
&= \lim_{R\to\infty} [0] \\
&= 0
\end{aligned}
$$

This looks remarkably similar to section 4.1. We saw a similar cancellation of the results of the upper and lower limits of integration. Let's look at graphically how this works. Imagine we are taking the integral of $g(x)$ on either side of the y-axis, moving out towards negative infinity and infinity where the edge of the filled in part of the graph is R on its path to approach to infinity .



Pictured above is the graph of $g(x) = \frac{2x}{x^2+1}$. The two graphs show $\lim_{R\to\infty} \int_{-R}^{R} g(x)\ dx$ as R is gradually getting larger and larger. It should be clear how the area in blue directly cancels with the area in red at any given R. This should provide a geometric intuition for why the Cauchy principal value is 0 in this case.

ii. Improper Integral

Now, we will try to calculate

$$\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\ dx$$

Recall, by definition 4.1,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\ dx &:= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \int_{\alpha}^{\beta} \frac{2x}{x^2+1}\ dx \\
&= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \left[\ln\left(\beta^2+1\right) - \ln\left(\alpha^2+1\right)\right] \\
&= \infty - \infty \\
&\Rightarrow \text{Indeterminate Form}
\end{aligned}$$

This is one way to look at the limit. We were left with something in indeterminate form, and thus should find another means of analyzing the limit. This could give us an intuition that the integral diverges and is something we will explore later. However, we need to show this another way.

From the intuition we just gained, we now claim that this limit diverges. Let's use the following proposition to prove this.

Proposition 5.1: *The following two statements are equivalent:*

1. *The improper integral exists* $\int_{-\infty}^{\infty} f(x)\ dx = L$.
2. *For any sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $\mathbb{R}$ *such that* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} b_n = \infty$, *then*

$$\lim_{n\to\infty} \int_{-a_n}^{b_n} f(x)\ dx = L$$

Let's suppose that α approaches negative infinity along some sequence $-\{a_n\}$ where $\{a_n\} \in \mathbb{R}$. Suppose that β approaches infinity along some sequence $k\{a_n\}$ where $k \in \mathbb{R}$ and positive . All of these approaches must yield the same result for the improper integral. We can translate these sequences in the following way. Consider below where $n, R \in \mathbb{R}$ and n is positive.

$$\lim_{R\to\infty} \int_{-R}^{nR} \frac{2x}{x^2+1}\ dx$$

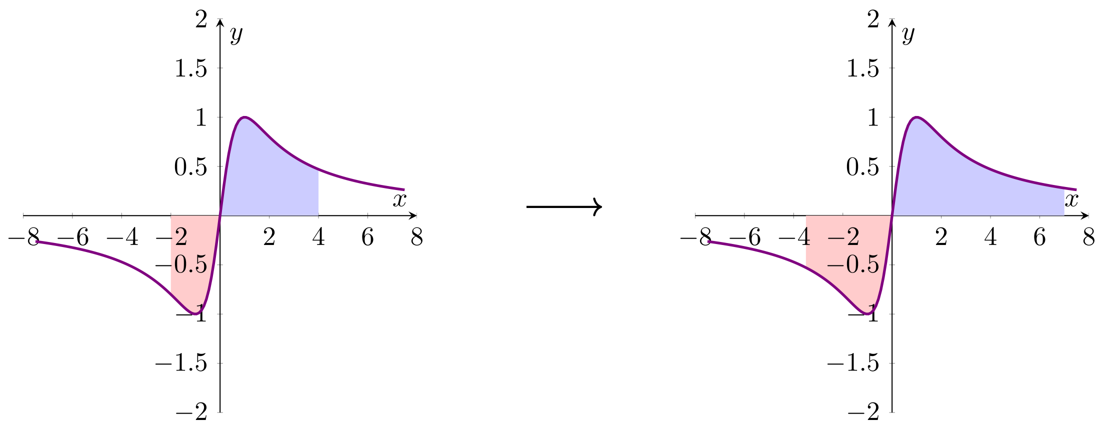
$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{nR}\frac{2x}{x^2+1}\,dx &= \lim_{R\to\infty}\left[\ln\left(x^2+1\right)\right]\Big|_{-R}^{nR}\\
&= \lim_{R\to\infty}\left[\ln\left(n^2R^2+1\right)-\ln\left((-R)^2+1\right)\right]\\
&= \lim_{R\to\infty}\left[\ln\left(\frac{n^2R^2+1}{R^2+1}\right)\right]\\
&= \ln\left[\lim_{R\to\infty}\left(\frac{n^2R^2+1}{R^2+1}\right)\right]\\
&= \ln\left[n^2\right]\\
&= n\ln(n)
\end{aligned}$$

As n is an arbitrary positive number, the value of the limit is clearly changing as the rate at which it approaches infinity changes. The limit is not constant and thus does not exist. As proposition 5.1 is a biconditional statement, we can thus say

$$\int_{-\infty}^{\infty}\frac{2x}{x^2+1}\,dx$$

does not exist.

As one last confirmation, lets look at $\lim_{R\to\infty}\int_{-R}^{nR}\frac{2x}{x^2+1}\,dx$ graphically where $n=2$.



It should be apparent that the difference between the volume on the right (in the blue) and that on the left (in the red) grow as R in $\lim_{R\to\infty}\int_{-R}^{nR}\frac{2x}{x^2+1}\,dx$ increases. This discrepancy will become unbounded as the limit is approached, meaning the integral will diverge.

We have thus found an example of a function where the Cauchy principal value disagrees with the improper integral (because it does not exist). This shows the importance of carefully following definitions and taking limits separately when dealing with integrals without bounded support. We can see how if the limits of the integrals exist separately, they will exist together–but not vice versa.

6 Generalization of Definition 1.5 (1.4 on Challenge Problem Report)

As a final section of our analysis of functions that are unbounded and do not have bounded support, we will attempt to generalize definition 1.4 to functions that are not continuous on the closed interval $[a,b]$ at a finite number of points but continuous elsewhere. Clearly, this will involve splitting up our integral a finite number of times, calculating them each individually, and putting them all together.

We will clearly need to sum our integrals between each $c_1<\cdots<c_k\subset[a,b]$. We also will need to evaluate from a to c_1 and from c_k to b. We know from single variable calculus that we can split up integrals by adjusting their limits of integration. Using this concept, splitting up our integrals will allow us to apply definition 1.5 repeatedly and sum them. Thus, our definition will look like the following

Definition 6.1: *Let $h(x)$ be a function that is unbounded at finitely many points $c_1<\cdots<c_k\subset[a,b]$, but is otherwise continuous on the interval $[a,b]$. If each of the following limits converge to a finite value, $\int_a^b h(x)\,dx$ is as*

follows. It is worth noting that if $k = 1$, the middle sigma term can be ignored and the limits to a and b on the edge terms can be replaced with the function value at their respective points.

$$\int_a^b h(x)\ dx := \lim_{\substack{\alpha\to a^+\\ \beta\to c_1^-}} \int_\alpha^\beta h(x)\ dx + \sum_{n=1}^{k-1}\left[\lim_{\substack{\alpha\to c_i^+\\ \beta\to c_{i+1}^-}} \int_\alpha^\beta h(x)\ dx\right] + \lim_{\substack{\alpha\to c_k^+\\ \beta\to b^-}} \int_\alpha^\beta h(x)\ dx$$

i. Testing Our Definition

We will now test this definition using a simple function. When we have a definition, the best way to unravel it is through concrete examples. Consider the function

$$g(x) = \frac{1}{x}$$

over the interval $[-1, 1]$.

By inspection, we can see that at $x = 0$, the function is undefined or unbounded. We can thus apply our definition because our function is unbounded at finitely many points (1 point) on this closed interval.

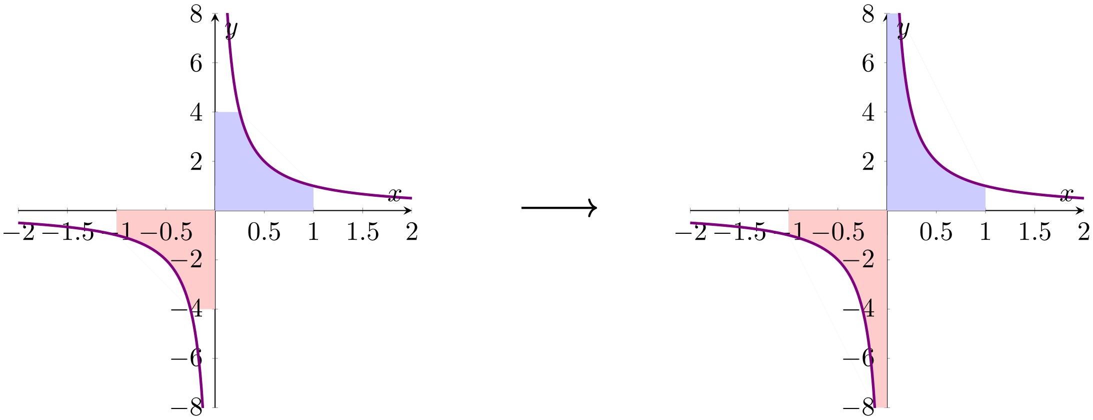
For reference, we will calculate the indefinite integral of $g(x)$.

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

Let's plug in.

$$\begin{aligned}
\int_a^b h(x)\ dx &:= \lim_{\substack{\alpha\to -1\\ \beta\to 0^-}} \int_\alpha^\beta g(x)\ dx + \sum_{n=1}^{0}\left[\lim_{\substack{\alpha\to c_i\\ \beta\to c_{i+1}}} \int_\alpha^\beta g(x)\ dx\right] + \lim_{\substack{\alpha\to 0^+\\ \beta\to 1}} \int_\alpha^\beta g(x)\ dx \\
&= \lim_{\substack{\alpha\to -1\\ \beta\to 0^-}} \int_\alpha^\beta g(x)\ dx + \lim_{\substack{\alpha\to 0^+\\ \beta\to 1}} \int_\alpha^\beta g(x)\ dx \\
&= \lim_{\beta\to 0^-} \int_{-1}^\beta g(x)\ dx + \lim_{\alpha\to 0^+} \int_\alpha^1 g(x)\ dx \\
&= \lim_{\beta\to 0^-} [\ln|\beta| - \ln|-1|] + \lim_{\alpha\to 0^+} [\ln(1) - \ln|\alpha|] \\
&= \lim_{\beta\to 0^-} [\ln|\beta|] + \lim_{\alpha\to 0^+} [-\ln|\alpha|] \\
&= -\infty + \infty \\
&\Rightarrow \text{This contradicts our definition but doesn't necessarily mean our improper integral does not exist}
\end{aligned}$$

Conveniently, our choice of $g(x)$ connects directly with section 1. We demonstrated that $\int_0^1 \frac{1}{x^p}\ dx$ where $p = 1$ does not exist. Thus, we could have skipped some of our above analysis and deduced that this indeterminate value would be the outcome. It is very important to note that while we end up with two single variable limits (to the same number-0-but from different sides) in this definition, they <u>cannot</u> be combined. Normally, we would try to treat a similar situation as a singular limit of multiple variables to demonstrate the need to independently approach a certain value but the nature of our definition requires them to be separate. Furthermore, we can connect with our discussion of the Cauchy principal value. Despite the fact that the Cauchy principal value is defined for functions that are continuous but lack bounded support, the idea of two different limits being required to exist when approaching at different rates applies very directly here. Let's look at our situation graphically like we did earlier for some context.



From the graphs, it is clear how these infinities will expand out and cancel each other out. However, it should also be clear how if the rates of approach to infinity were different, the difference in between the integrals would approach infinity. We will do a brief similar analysis choosing the same paths (to different values however) using $\lim_{R\to 0}\left[\int_{-1}^{-R}\frac{1}{x}\ dx+\int_{nR}^{1}\frac{1}{x}\ dx\right]$. We would again see how this would not be a constant value, and thus, this limit does not exist.

$$\begin{aligned}\lim_{R\to 0}\left[\int_{-1}^{-R}\frac{1}{x}\ dx+\int_{nR}^{1}\frac{1}{x}\ dx\right] &= \lim_{R\to 0}\left[(\ln|x|)\,|_{-1}^{-R}+(\ln|x|)\,|_{nR}^{1}\right]\\ &= \lim_{R\to 0}\left[\ln(1)-\ln|nR|+\ln|R|-\ln(1)\right]\\ &= \lim_{R\to 0}\ln|\frac{1}{n}|\\ &= -\ln|n|\end{aligned}$$

Taken together, we see that our definition works as this integral should not be defined. But, we most importantly were reminded of the need to consider points of unboundedness individually–whether that is a function being unbounded or a function not having bounded support. It is crucial not to get caught in the trap of taking something like the Cauchy principal value when evaluating an improper integral.

7 Food For Thought

In this section, we will briefly explain the answers to the questions in the section referencing some new theorems. Then we will attempt to expand on a final theorem in certain conditions. Hopefully, this will connect with some of our earlier anaylsis in sections 4 and 5.

(A) Consider the function: $f(x)=xe^{-x^2}$

(1) $\lim_{\alpha\to\infty}\left(\int_{-\alpha}^{0}f(x)\ dx\right)$

Recall that we worked with this function earlier. We know the antiderivative is

$$\int f(x)\ dx=-\frac{1}{2}e^{-x^2}+C$$

Now consider:

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left(\int_{-\alpha}^{0} xe^{-x^2}\ dx\right)\\
&= \lim_{\alpha\to\infty}\left[\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_{-\alpha}^{0}\right]\\
&= -\frac{1}{2}e^{-0^2} - \lim_{\alpha\to\infty}\left(-\frac{1}{2}e^{-\alpha^2}\right)\\
&= -\frac{1}{2} - 0\\
&= \boxed{-\frac{1}{2}}
\end{aligned}$$

(2) $\lim_{\beta\to\infty}\left(\int_0^\beta f(x)\ dx\right)$
This will reduce very similarly to above as

$$\begin{aligned}
&\lim_{\beta\to\infty}\left(-\frac{1}{2}e^{-\beta^2}\right) - \left(-\frac{1}{2}e^{-0^2}\right)\\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(3) Let's make a conclusion about $\int_{-\infty}^{\infty} xe^{-x^2}\ dx$ and its relation to $P.V. \int_{-\infty}^{\infty} xe^{-x^2}\ dx$.
We need the following theorems.

Theorem 7.1: *If $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathbb{R}$, then if both $\lim_{n\to\infty} f_n$ and $\lim_{n\to\infty} g_n$ exist (e.g converge), then*

$$\lim_{n\to\infty} f_n + \lim_{n\to\infty} g_n = \lim_{n\to\infty}(f_n + g_n)$$

Theorem 7.2: *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $\mathbb{R}$. If $\lim_{\alpha\to-\infty}\left(\int_\alpha^0 f(x)\ dx\right) = A$ exists, and $\lim_{\beta\to\infty}\left(\int_0^\beta f(x)\ dx\right) = B$ exists, then the improper integral $\int_{-\infty}^{\infty} f(x)\ dx$ exists and*

$$\int_{-\infty}^{\infty} f(x)\ dx = A + B$$

By Theorem 7.2, we can conclude that $\int_{-\infty}^{\infty} xe^{-x^2}\ dx$ is the sum of the two integrals we just calculated $\Rightarrow 0$. As calculated and compared in section 4, this agrees with the Cauchy principal value.

(B) Consider the function: $g(x) = \frac{2x}{x^2+1}$
Reference section 5; we already calculated the indefinite integral to be: $\int g(x)\ dx = \frac{2x}{x^2+1}\ dx = \ln\left(x^2+1\right) + C$
(1) $\lim_{\alpha\to\infty}\left(\int_{-\alpha}^0 g(x)\ dx\right)$
We can calculate this quickly from what we have done so far.

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left(\int_{-\alpha}^{0} \frac{2x}{x^2+1}\ dx\right)\\
&= \ln\left(0^2+1\right) - \lim_{\alpha\to\infty}\left[\ln\left(\alpha^2+1\right)\right]\\
&= \ln(1) - \infty\\
&= \boxed{-\infty}
\end{aligned}$$

(2) $\lim_{\beta\to\infty}\left(\int_0^\beta g(x)\ dx\right)$
Similarly, this produces

$$\begin{aligned}&\lim_{\beta\to\infty}\left(\int_0^\beta \frac{2x}{x^2+1}\ dx\right)\\ =&\lim_{\beta\to\infty}\left[\ln\left(\beta^2+1\right)\right]-\ln\left(0^2+1\right)\\ =&\ \infty-\ln(1)\\ =&\ \boxed{\infty}\end{aligned}$$

(3) Conclusion

We now want to conclude that $\int_{-\infty}^{\infty}\frac{2x}{x^2+1}\ dx \neq P.V.\int_{-\infty}^{\infty}\frac{2x}{x^2+1}\ dx$. We should have an overwhelming intuition that this is a correct statement. However, as the limits don't exist, theorem 7.2 does not apply, and we can thus not use that to make any conclusion. We again would need to consider the limit at different sequences approaching infinity. This would require some integral with bounds R and nR as we did earlier. It should be clear by now that we would end up with different results. Let's look into why this will always be the case if our individual limits approach infinity.

Essentially, we want to be able to skip this path analysis for every problem we encounter. However, tackling this problem relies heavily on our definition of limits not existing. For parts of this challenge report, we considered any divergence to infinity or negative infinity of an integral as "not existing." Section 1 is a good example of this. In the following section, we will note the difference between $\pm\infty$ and a limit not existing. This is how we have specified our answers so far in this section. We claim to have the following

Proposition 7.3: Expanding on Definition 7.2

Let $f:\mathbb{R}\to\mathbb{R}$ be a function that is continuous on $\mathbb{R}$. If $\lim_{\alpha\to-\infty}\left(\int_\alpha^0 f(x)\ dx\right)=\pm\infty$, and $\lim_{\beta\to\infty}\left(\int_0^\beta f(x)\ dx\right)=\mp\infty$, then the improper integral $\int_{-\infty}^{\infty} f(x)\ dx$ does not exist.

Put another way, if the improper integral on one side diverges to either positive or negative infinity and if the improper integral on the other end of the $x-$axis diverges to the opposite infinity, the improper integral does not exist–meaning again that it is not defined to be infinity or negative infinity.

This proposition would allow us to definitively say after solely seeing that results in B1 and B2, the improper integral $\int_{-\infty}^{\infty} g(x)\ dx$ does not exist. We won't write a proof but will explain the intuition. Consider a function $f:\mathbb{R}\to\mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{\substack{\beta\to\infty\\ \alpha\to-\infty}}\int_\alpha^\beta f(x)\ dx$$

We can always define a path for how α and β relate to one another. This path can change and the limit must remain the same. We have used this idea many times and is the intuition beyond our above proposition.

Summary

In this report, we developed a thorough understanding of the needed precautions in dealing with integrals of functions that are unbounded and/or do not have bounded support. It is possible to do, but requires specific, careful steps. First, we looked at a function that was unbounded. Then, we looked at unbounded support in sections 2-5. Finally, we connected back to unbounded functions where we analyzed them in a new light from what we had discovered about functions without bounded support. In order to put this report together, I checked some of my answers with Jack Hambidge, Lucas Schardt, Hero Jay, and Bogdan Yaremenko. I took a few ideas and checked my answers in office hours. I also clarified some of my understanding in office hours with Professor Wong of limits of single compared to multiple variables and what it means for an integral to not exist. I relied on both the Challenge Problem Report question document and the lecture notes for definitions/theorems/propositions.