# Challenge Problem Report 3

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This Challenge Problem Report explores the Frenet frame and curvature, specifcally how these behave with regard to vector-valued functions from $\mathbb{R} \mapsto \mathbb{R}^3$. We will first develop a clear understanding of what the Frenet frame is and its basic properties. We will build on these properties to consider what $\boldsymbol{r}''(t)$ is made up of. We will use examples along the way to check and expand understanding. Then, we will dive into curvature and how it generalizes through an osculating circle, ending by finding an alternative formula for curvature. The Frenet frame is important in many fields beyond mathematics as it is vector-valued functions that can describes curves in 3d space, which has clear practical application to the real world. It is worth developing a thorough understanding of these ideas.

## 1 Showing a Frenet Frame is an Orthonormal Basis

Our goal in this section will be to establish what a Frenet frame is and show that it consistently provides an orthonormal basis for $\mathbb{R}^3$. The implications of the Frenet frame providing an orthonormal basis for $\mathbb{R}^3$ are critical to understanding how a Frenet frame functions. Proving this orthogonality is crucial in understanding the geometric meaning of the Frenet frame.

First and foremost, we need to establish what a Frenet frame is. We will use the below definition. This definition will assume that we start with the vector-valued function $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$.

**Definition 1.1:** *The Frenet frame is the collection of vectors* $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ *where* $\boldsymbol{T}, \boldsymbol{N}$, *and* $\boldsymbol{B}$ *are defined as follows:*

$$\boldsymbol{T}(t) = \frac{1}{||\boldsymbol{r}'(t)||}\boldsymbol{r}'(t)$$

$$\boldsymbol{N}(t) = \frac{1}{||\boldsymbol{T}'(t)||}\boldsymbol{T}'(t)$$

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$$

We must also consider the definition of a basis if we plan to show that the Frenet frame is an orthonormal basis.

**Definition 1.2:** *An ordered set of vectors* $\mathcal{B}$ *is a basis of* $V$ *if*

1. $\mathcal{B} \subset V$
2. $\text{span}(\mathcal{B}) = V$
3. $\mathcal{B}$ is linearly independent

*For a basis to be orthonormal, all the components of* $\mathcal{B}$ *must be orthogonal to each other and have a magnitude of 1.*

However, it should be noted that these are not the conditions that we will directly show to prove that the Frenet frame is an orthonormal basis. We primarily will show that the three vectors contained in the Frenet frame are orthonormal, indirectly proving the above conditions. Three orthonormal vectors in $\mathbb{R}^3$ are by definition linearly independent. Furthermore, because there are 3 vectors and they are linearly independent, they will be able to span $\mathbb{R}^3$. Perhaps it is worth noting that $n$ linearly independent vectors in $\mathbb{R}^n$ must be able to span $\mathbb{R}^n$. We will now move on to proving that the three vectors in the Frenet frame are orthonormal which will thus prove the Frenet frame is an orthonormal basis. See the following

**Proof:**

We want to show that because $\boldsymbol{T}(t)$ has a constant length, $\boldsymbol{T}'(t)$ is orthogonal to $\boldsymbol{T}(t)$. This will provide the crux of the proof as a Frenet frame by definition is a collection of vectors with a constant magnitude, a normalized magnitude of 1. By the definition of the dot product, if we show that some $\boldsymbol{T}(t) \cdot \boldsymbol{T}'(t) = 0$, we know that $\boldsymbol{T}$ and $\boldsymbol{N}$ will be orthogonal as we can simply factor out the scalar $||\boldsymbol{T}'(t)||$ in our analysis. Then, by the magnitude formula for the cross product, we will show that all three vectors will be mutually orthonormal. This is sufficient to prove the Frenet frame is an orthonormal basis.

Consider the following knowing that $||\boldsymbol{T}(t)||$ is constant:

$$||\boldsymbol{T}(t)||$$

and

$$\boldsymbol{T}(t) \cdot \boldsymbol{T}(t)$$

By the definition of the dot product, we know that

$$||\boldsymbol{T}(t)||^2 = \boldsymbol{T}(t) \cdot \boldsymbol{T}(t)$$

Taking the derivative with respect to $t$ on both sides produces:

$$\frac{d}{dt}||\boldsymbol{T}(t)||^2 = \frac{d}{dt}\left(\boldsymbol{T}(t) \cdot \boldsymbol{T}(t)\right)$$

Because the derivative of a constant is 0 and there exists a product rule for the dot product, we have the following:

$$0 = \boldsymbol{T}'(t) \cdot \boldsymbol{T}(t) + \boldsymbol{T}(t) \cdot \boldsymbol{T}'(t)$$

$$0 = 2\left(\boldsymbol{T}'(t) \cdot \boldsymbol{T}(t)\right)$$

$$0 = \boldsymbol{T}'(t) \cdot \boldsymbol{T}(t)$$

We have thus proven that $\boldsymbol{T}'(t)$ is orthogonal to $\boldsymbol{T}(t)$ as $||\boldsymbol{T}(t)||$ is always 1. We need to take this a step further to prove that $\boldsymbol{T}(t)$ is orthogonal to $\boldsymbol{N}(t)$. We will rearrange our formula in definition 1.1 to produce an $\boldsymbol{N}$. Consider our formula

$$\begin{gathered}\boldsymbol{N}(t) = \frac{1}{||\boldsymbol{T}'(t)||}\boldsymbol{T}'(t) \\ \boldsymbol{N}(t)||\boldsymbol{T}'(t)|| = \boldsymbol{T}'(t) \\ \boldsymbol{T}'(t) = \boldsymbol{N}(t)||\boldsymbol{T}'(t)||\end{gathered}$$

We can easily substitute in our modified formula for $\boldsymbol{T}'(t)$ into our previous analysis. We know that $||\boldsymbol{T}'(t)||$ is a scalar. Thus, we can divide it out, leaving us with our desired result and showing orthogonality.

$$\begin{gathered}0 = \boldsymbol{T}'(t) \cdot \boldsymbol{T}(t) \\ 0 = \boldsymbol{N}(t)||\boldsymbol{T}'(t)|| \cdot \boldsymbol{T}(t) \\ 0 = \boldsymbol{N}(t) \cdot \boldsymbol{T}(t)\end{gathered}$$

This analysis is useful because $\boldsymbol{T}$ has a constant length and will thus always be orthogonal to $\boldsymbol{N}$. Furthermore, by definition, $\boldsymbol{N}$ is also normalized to have a magnitude of 1. Thus, we have two of our three vectors for our orthonormal basis.

Now, lets consider $\boldsymbol{B}$. Note, $\boldsymbol{B}$ is defined as follows:

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$$

By the definition of the cross product in $\mathbb{R}^3$, we know that two orthogonal vectors crossed with each other will produce a third vector that is orthogonal to both vectors. However, we are looking for an orthonormal basis. Essentially, all that is then left to prove is that the magnitude of $\boldsymbol{B}$ is 1. Let us use a formula for the cross product to show this.

**Definition 1.3:** *Suppose vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. Suppose an angle $\theta$ is the measure of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The magnitude of their cross product is defined by the following formula:*

$$||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}||||\boldsymbol{b}|| \sin(\theta)$$

We know from our earlier analysis that $\boldsymbol{T}$ and $\boldsymbol{N}$ are orthogonal so the angle between them, $\theta$, is $\frac{\pi}{2}$. Using the formula immediately above, we can see that $||\boldsymbol{B}||$ is equal to the following:

$$= ||\boldsymbol{T}||||\boldsymbol{N}|| \sin(\theta)$$

$$= (1)(1) \sin(\frac{\pi}{2})$$

$$= 1$$

Thus, we have proven that $\boldsymbol{B}$ has a magnitude of 1. $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ all have magnitudes of 1 and are all orthogonal.

Taken together, we can see that the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ is made up of three orthonormal vectors. As described earlier, three orthonormal vectors are both linearly independent and span $\mathbb{R}^3$, fulfilling the definition of an orthonormal basis.

Thus, $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ is an orthonormal basis. $\square$

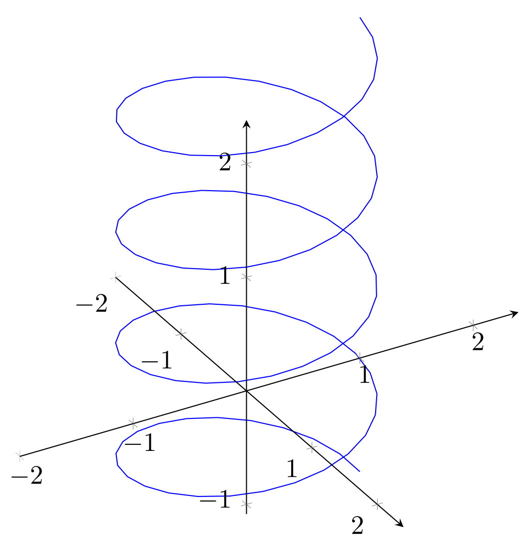
## 2 Finding the Frenet Frame for some $\vec{r}(t)$

In the following section, we will take our understanding of the Frenet frame from section 1 and apply it to a specific example. This will illustrate a more geometric picture of the meaning of the Frenet frame, allowing us to see a visual behind what we just proved. Furthermore, we will be able to see how we can put our above formulas to use. The Frenet frame, as explained in the introduction, is critical in areas beyond the math world. Thus, putting it to use on examples is important.

In this section, we will therefore consider the following vector-valued function:

$$\boldsymbol{c}(t) = \langle -\sin(t), \cos(t), t \rangle$$

First, we will sketch this function. The arrows showing the path of the function will be shown on later graphs. Essentially, because the $z$-component is the only non trigonometric function, $\boldsymbol{c}(t)$ should circle upwards around the $z$-axis *Note: the code to sketch this function was modified from Dr. Wong's LaTeXfile*




We will now gradually develop our Frenet frame. In our first part, we will find two values that are critical to solve for to come up with our Frenet frame. Then, we will finish solving for our Frenet frame in our second part.

**i. Computing and Sketching $\boldsymbol{c}'(\frac{\pi}{2})$ and $\boldsymbol{T}'(\frac{\pi}{2})$**

We will compute then sketch these two vectors because of their relation to both $\boldsymbol{T}$ and $\boldsymbol{N}$, two of the three components of the Frenet frame. From the formulas in definition 1.1, we can see that $\boldsymbol{T}(t)$ is just $\boldsymbol{c}'(t)$ normalized and the same is true with $\boldsymbol{T}'(t)$ and $\boldsymbol{N}(t)$.

Let us attempt to find these values noting the following formula. We can take the derivative of a vector-valued function from $\mathbb{R} \to \mathbb{R}^n$ componentwise.

**Theorem 2.1:** *The derivative of a function $\boldsymbol{r}(t) : \mathbb{R} \to \mathbb{R}^n$ at an interior point $\boldsymbol{t_0}$ is the linear transformation $T : \mathbb{R} \to \mathbb{R}^n$ where*

$$T = \begin{bmatrix} x_1'(t_0) \\ . \\ . \\ . \\ x_n'(t_0) \end{bmatrix}$$

We will use this idea to take the derivative of $\boldsymbol{c}(t)$. We know how to take these derivatives from single variable calculus. Observe the following

$$\boldsymbol{c}(t) = \langle -\sin(t), \cos(t), t \rangle$$

Thus,

$$\boldsymbol{c}'(t) = \langle -\cos(t), -\sin(t), 1 \rangle$$

Now, we have found $\boldsymbol{c}'(t)$. This vector-valued function, evaluated at $\frac{\pi}{2}$, is one of the two vectors we are looking for. Using this function, we can go further to find this other vector. We will now use our equations in definition 1.1 to calculate $\boldsymbol{T}(t)$.

$$\boldsymbol{T}(t) = \frac{1}{||\boldsymbol{c}'(t)||}\boldsymbol{c}'(t)$$

Clearly, we need to find the magnitude of $\boldsymbol{c}'(t)$. We can use Pythagorean Theorem to do this:

$$\begin{aligned} ||\boldsymbol{c}'(t)|| &= \sqrt{\cos^2(t) + \sin^2(t) + 1} \\ &= \sqrt{1+1} \\ &= \sqrt{2} \end{aligned}$$

Plugging in, we see that $\boldsymbol{T}(t)$ is normalized to be:

$$\boldsymbol{T}(t) = \left\langle \frac{-\cos(t)}{\sqrt{2}}, \frac{-\sin(t)}{\sqrt{2}}, \frac{\sqrt{2}}{2} \right\rangle$$

We have thus found $\boldsymbol{T}(t)$ using $\boldsymbol{c}'(t)$ and then normalizing it by dividing by its magnitude. We just have one more derivation step to go before we evaluate at $\frac{\pi}{2}$.

Once again, we simply need to derive componentwise to determine $\boldsymbol{T}'(t)$. Thus,

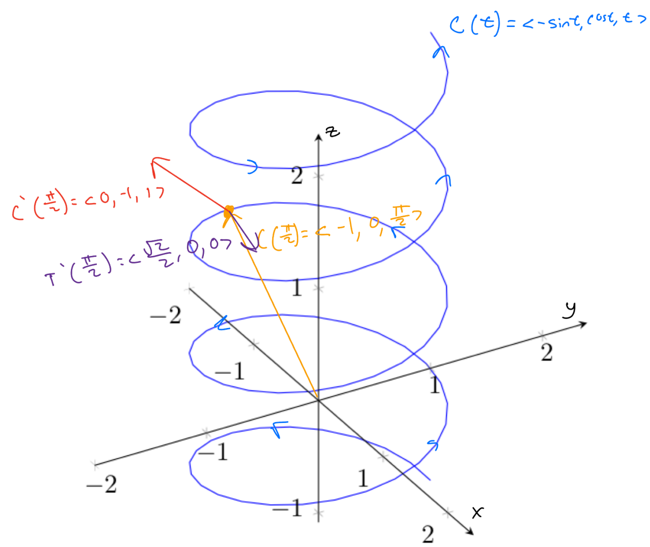
$$\boldsymbol{T}'(t) = \left\langle \frac{\sin(t)}{\sqrt{2}}, \frac{-\cos(t)}{\sqrt{2}}, 0 \right\rangle$$

We want the specific values of $\boldsymbol{c}'(t)$ and $\boldsymbol{T}'(t)$ at $t = \frac{\pi}{2}$. Here are the results:

$$\begin{aligned} \boldsymbol{c}'(t) &= \langle -\cos(t), -\sin(t), 1 \rangle \\ \boldsymbol{c}'\left(\frac{\pi}{2}\right) &= \langle -\cos\left(\frac{\pi}{2}\right), -\sin\left(\frac{\pi}{2}\right), 1 \rangle \\ &= \langle 0, -1, 1 \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{T}'(t) &= \left\langle \frac{\sin(t)}{\sqrt{2}}, \frac{-\cos(t)}{\sqrt{2}}, 0 \right\rangle \\ \boldsymbol{T}'\left(\frac{\pi}{2}\right) &= \left\langle \frac{\sin(\frac{\pi}{2})}{\sqrt{2}}, \frac{-\cos(\frac{\pi}{2})}{\sqrt{2}}, 0 \right\rangle \\ &= \left\langle \frac{\sqrt{2}}{2}, 0, 0 \right\rangle \end{aligned}$$

Now, we will sketch both of our vectors, starting at the value of $\boldsymbol{c}(\frac{\pi}{2}) = \langle -1, 0, \frac{\pi}{2} \rangle$. Here is the sketch of both $\boldsymbol{T}'(\frac{\pi}{2})$ and $\boldsymbol{c}'(\frac{\pi}{2})$.




**ii. The Full Frenet Frame for $\boldsymbol{c}(t)$**

Now that we have found both $\boldsymbol{T}'(\frac{\pi}{2})$ and $\boldsymbol{c}'(\frac{\pi}{2})$, we simply need to normalize both of these vectors to produce $\boldsymbol{N}(t)$ and $\boldsymbol{T}(t)$ respectively. Then, we can cross these vectors to form our frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. Lastly, we will plug in $t = \frac{\pi}{2}$.

However, in finding $\boldsymbol{T}'(t)$, we already found $\boldsymbol{T}(t)$.

$$\boldsymbol{T}(t) = \left\langle \frac{-\cos(t)}{\sqrt{2}}, \frac{-\sin(t)}{\sqrt{2}}, \frac{\sqrt{2}}{2} \right\rangle$$

$$\boldsymbol{T}\left(\frac{\pi}{2}\right) = \left\langle 0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

Thus, let us solve for $\boldsymbol{N}(t)$ by normalizing $\boldsymbol{T}'(t)$. From definition 1.1, that is all we need to do. We will use our results from part i. Consider the following

$$||\boldsymbol{T}'(t)||$$

$$\begin{aligned} \left|\left|\boldsymbol{T}'\left(\frac{\pi}{2}\right)\right|\right| &= \left|\left|\left\langle \frac{\sin(\frac{\pi}{2})}{\sqrt{2}}, \frac{-\cos(\frac{\pi}{2})}{\sqrt{2}}, 0 \right\rangle\right|\right| \\ &= \left|\left|\left\langle \frac{\sqrt{2}}{2}, 0, 0 \right\rangle\right|\right| \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

After finding the magnitude of $\boldsymbol{T}'(t)$, we can divide $\boldsymbol{T}'(t)$ by this magnitude. We can distribute the scalar into the vector $\boldsymbol{T}'(t)$. It will then produce that:

$$\boldsymbol{N}\left(\frac{\pi}{2}\right) = \langle 1, 0, 0 \rangle$$

Finally, we only have $\boldsymbol{B}(\frac{\pi}{2})$ left to solve for. By our formula, we will simply calculate $\boldsymbol{T}(\frac{\pi}{2}) \times \boldsymbol{N}(\frac{\pi}{2})$. Observe:

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$$

$$\boldsymbol{B}\left(\frac{\pi}{2}\right) = \left\langle 0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \times \langle 1, 0, 0 \rangle$$

From our knowledge of the cross product, we know that we can calculate it using a determinant in the following way. Observe:

$$\boldsymbol{B}\left(\frac{\pi}{2}\right) = \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}\right)$$

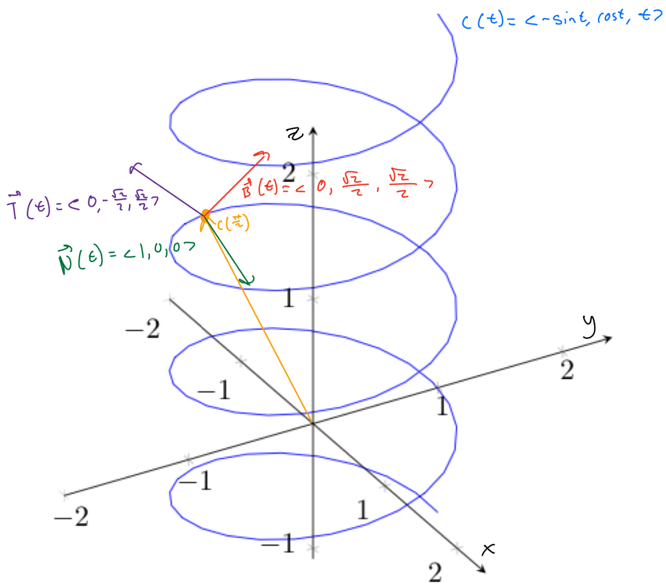
$$\boldsymbol{B}\left(\frac{\pi}{2}\right) = \langle \det\left(\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix}\right), \det\left(\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ 1 & 0 \end{bmatrix}\right), \det\left(\begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} \\ 1 & 0 \end{bmatrix}\right)\rangle$$

$$\boldsymbol{B}\left(\frac{\pi}{2}\right) = \langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$$

Now that we have successfully calculated $\boldsymbol{B}\left(\frac{\pi}{2}\right)$, we have all the elements of our Frenet frame. They are as follows:

$$\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\} = \{\langle 0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle, \langle 1, 0, 0\rangle, \langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle\}$$

As a final step, we shall again sketch the graph of $\boldsymbol{c}(t)$. Then, we will sketch the corresponding Frenet frame at $t = \frac{\pi}{2}$ at the point $\boldsymbol{c}(\frac{\pi}{2})$. Hopefully, this sketch will help to illustrate our proof in section 1 geometrically. It should be clear that these three vectors orthogonal, showing the results of our proof are correct. The sketch is the following:




## 3 Proof that the Acceleration Vector is a Linear Combination of $\vec{N}$ and $\vec{T}$

We have now developed both an algebraic and geometric understanding of the Frenet frame. However, there are many more relationships between the various elements of the Frenet frame and their properties that need to be explored.

In general, suppose we are working with a vector-valued function $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$. We can clearly see that $\boldsymbol{T}(t)$ is parallel to $\boldsymbol{r}'(t)$, meaning that $\boldsymbol{T}(t) = \lambda \boldsymbol{r}'(t)$. At first glance, it may seem as if the same is true for $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$. Because of how their definitions in definition 1.1 are not directly linked, the chain rule and product rule make it so this is not the case for all $\boldsymbol{r}(t)$. However, we can show that $\boldsymbol{r}''(t)$ is a linear combination of $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$. This means that $\boldsymbol{r}''(t)$ will lie in something called the osculating plane as it is the combination of two vectors. We will go on to see in later sections various other the implications of this result.

In order to prove that the acceleration vector $\boldsymbol{r}''(t)$ is a linear combination of $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$, we need to first define what it means for something to be a linear combination. Consider the following definition.

**Definition 3.1:** *Suppose $V$ is a vector space. A linear combination of $k$ vectors $\boldsymbol{v_1}, ..., \boldsymbol{v_k} \in V$ is some vector in the following form*

$$\lambda_1 \boldsymbol{v_1} + ... + \lambda_k \boldsymbol{v_k}$$

*where $\lambda_i$ are scalars.*

**Proof:**

We want to show that $\boldsymbol{r}''(t) = \alpha \boldsymbol{N}(t) + \beta \boldsymbol{T}(t)$ where $\alpha, \beta \in \mathbb{R}$.

In order to start our proof, we clearly need to find a formula for $\boldsymbol{r}''(t)$. We will do this by starting with our formula in definition 1.1 for $\boldsymbol{T}(t)$, moving around our scalar, and taking the derivative of both sides with respect to $t$.

$$\begin{aligned}
\boldsymbol{T}(t) &= \left|\left|\frac{1}{\boldsymbol{r}'(t)}\right|\right| \boldsymbol{r}'(t) \\
\boldsymbol{r}'(t) &= \boldsymbol{T}(t) \, ||\boldsymbol{r}'(t)|| \\
\boldsymbol{r}''(t) &= \boldsymbol{T'}(t) \, ||\boldsymbol{r}'(t)|| + \boldsymbol{T}(t) \frac{d}{dt} \, ||\boldsymbol{r}'(t)||
\end{aligned}$$

Now, we will pause to consider how we will calculate $\frac{d}{dt}||\boldsymbol{r}'(t)||$. Determining this will allow us to progress further in what we want to show. From our want-to-show statement, we need to find a way to make $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ appear on the right hand side. $\boldsymbol{T}(t)$ is already present, so we will focus in shortly on producing $\boldsymbol{N}(t)$

$$\begin{aligned}
\frac{d}{dt} \, ||\boldsymbol{r}'(t)|| &= \frac{d}{dt} \sqrt{\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(t)} \\
&= \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t) + \boldsymbol{r}''(t) \cdot \boldsymbol{r}'(t)}{2\sqrt{\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(t)}} \\
&= \frac{2\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{2\sqrt{\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(t)}} \\
&= \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{\sqrt{\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(t)}} \\
&= \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||}
\end{aligned}$$

We will now continue on, plugging in our derivative to progress further. We will cleverly multiply the top and bottom of the first term on the right side of our equals sign by $||\boldsymbol{T'}(t)||$ to produce $\boldsymbol{N}(t)$ once again using definition 1.1.

$$\begin{aligned}
\boldsymbol{r}''(t) &= \boldsymbol{T'}(t) \, ||\boldsymbol{r}'(t)|| + \boldsymbol{T}(t) \frac{d}{dt} \, ||\boldsymbol{r}'(t)|| \\
&= \boldsymbol{T'}(t) \, ||\boldsymbol{r}'(t)|| + \boldsymbol{T}(t) \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||} \\
&= \frac{\boldsymbol{T'}(t) \, ||\boldsymbol{r}'(t)|| \, ||\boldsymbol{T'}(t)||}{||\boldsymbol{T'}(t)||} + \boldsymbol{T}(t) \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||} \\
&= \boldsymbol{N}(t) \, ||\boldsymbol{r}'(t)|| \, ||\boldsymbol{T'}(t)|| + \boldsymbol{T}(t) \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||}
\end{aligned}$$

Notice how only $\boldsymbol{N}(t)$ and $\boldsymbol{T}(t)$ are vectors in our above formula for $\boldsymbol{r}''(t)$. Everything else is a scalar. We will assign an $\alpha$ and a $\beta$ to represent the scalars to more clearly recreate the definition of a linear combination.

$$\alpha = ||\boldsymbol{r}'(t)|| \, ||\boldsymbol{T'}(t)|| \text{ and } \beta = \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||}$$

We can rewrite our last line in our above analysis and see the following.

$$\boldsymbol{r}''(t) = \alpha \boldsymbol{N}(t) + \beta \boldsymbol{T}(t)$$

From the definition of a linear combination and as $\boldsymbol{r}''(t)$ is the sum of the product of scalars and $\boldsymbol{N}(t)$ and $\boldsymbol{T}(t)$ respectively, clearly $\boldsymbol{r}''(t)$ is a linear combination of $\boldsymbol{N}(t)$ and $\boldsymbol{T}(t)$. $\square$

# 4 Comparing $\vec{N}(t)$ and $\vec{r}''(t)$

As mentioned at the beginning of section 3, at quick first glance, it may seem like $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are parallel. We very clearly proved that this is not always the case as $\boldsymbol{r}''(t)$ may also be made up of $\boldsymbol{T}(t)$ from our result in section 3. Nevertheless, it is worth looking at cases where $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are parallel as this is certainly a possibility if the coefficient of the term $\boldsymbol{T}(t)$ in our above result is 0. Similar to our goal of understanding section 1 through an example in section 2, we can use examples to explore our results in section 3. This should further build understanding of the Frenet frame.

In our first subsection, we will consider an example of an earlier vector-valued function we analyzed in an attempt to show an instance where $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are parallel. In our second subsection, we will consider another example from a section that will come up later in our analysis where $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are not parallel.

**i. $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are parallel.**

Recall our function from section 2,

$$\boldsymbol{c}(t) = \langle -\sin(t), \cos(t), t \rangle$$

We saw that $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ were parallel by a factor of $\sqrt{2}$ at $t = \frac{\pi}{2}$. Let us show that it is parallel for all $t$. This means ensuring that the value of the coefficient we calculated in section 3 for $\boldsymbol{T}(t)$ is zero.

$$\frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||} = 0$$

Let us now copy what we discovered about this function in section 2.

$$\begin{aligned} \boldsymbol{c}'(t) &= \langle -\cos(t), -\sin(t), 1 \rangle \\ \boldsymbol{T}(t) &= \left\langle \frac{-\cos(t)}{\sqrt{2}}, \frac{-\sin(t)}{\sqrt{2}}, \frac{\sqrt{2}}{2} \right\rangle \\ \boldsymbol{T}'(t) &= \left\langle \frac{\sin(t)}{\sqrt{2}}, \frac{-\cos(t)}{\sqrt{2}}, 0 \right\rangle \end{aligned}$$

Recall in section 2, we were evaluating this equation at a specific point. We were able to begin plugging in said point after these steps. Recall that we are looking for both $\boldsymbol{c}''(t)$ and $\boldsymbol{N}(t)$. We only have a little bit further to go to get both of these values.

For $\boldsymbol{c}''(t)$, we can simply derive $\boldsymbol{c}'(t)$ componentwise according to theorem 2.1.

$$\begin{aligned} \boldsymbol{c}'(t) &= \langle -\cos(t), -\sin(t), 1 \rangle \\ \boldsymbol{c}''(t) &= \langle \sin(t), -\cos(t), 0 \rangle \end{aligned}$$

For $\boldsymbol{N}(t)$, we just need to normalize $\boldsymbol{T}'(t)$ by finding its magnitude by the following:

$$\begin{aligned} ||\boldsymbol{T}'(t)|| &= \sqrt{\frac{\sin^2(t)}{2} + \frac{\cos^2(t)}{2} + 0} \\ &= \sqrt{\frac{1}{2}\left(\sin^2 + \cos^2\right)} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

To normalize $\boldsymbol{T}'(t)$ according to our formula in definition 1.1, we need to divide by $||\boldsymbol{T}'(t)||$. Or equivalently, multiply by its reciprocal, $\sqrt{2}$, like so

$$\begin{aligned}\boldsymbol{N}(t) &= \frac{1}{||\boldsymbol{T}'(t)||}\boldsymbol{T}'(t) \\ &= \sqrt{2}\left\langle \frac{\sin(t)}{\sqrt{2}}, \frac{-\cos(t)}{\sqrt{2}}, 0 \right\rangle \\ &= \langle \sin(t), -\cos(t), 0\rangle\end{aligned}$$

Now we have values for both $\boldsymbol{N}(t)$ and $\boldsymbol{c}''(t)$. They are not only parallel, but in this case they are the same vector! We have previously mentioned that to be parallel means to be some scalar multiple of each other. In this case, that scalar multiple is 1.

$$\boldsymbol{N}(t) = \boldsymbol{c}''(t) = \langle \sin(t), -\cos(t), 0\rangle$$

**ii. $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are not parallel.**

We just observed a vector-valued function such that the second derivative of the function itself was parallel to $\boldsymbol{N}(t)$. This meant making the coefficient in the linear combination for $\boldsymbol{r}''(t)$ of $\boldsymbol{T}(t)$ equal to 0. Now, we want

$$\frac{\boldsymbol{r}'(t)\cdot \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t)||} \neq 0$$

We are just looking for an example of a vector-valued function. In general, our example in part i is one that contains simple functions with relatively easy derivatives, and more importantly, many of our key vectors have constant magnitudes. Let us consider the following function that we will address in more depth later.

$$\boldsymbol{r}(t) = \langle \ln(t), 1, t\rangle$$

Remember, our goal is to find both $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$. The algebra will be slightly more arduous in this subsection. Let's focus on $\boldsymbol{r}''(t)$ first by deriving componentwise twice. Then, we can focus on $\boldsymbol{N}(t)$.

$$\begin{aligned}\boldsymbol{r}(t) &= \langle \ln(t), 1, t\rangle \\ \boldsymbol{r}'(t) &= \langle \frac{1}{t}, 0, 1\rangle \\ \boldsymbol{r}''(t) &= \langle -\frac{1}{t^2}, 0, 0\rangle\end{aligned}$$

Okay, we will now start the algebra to find $\boldsymbol{N}(t)$. A important first step is to calculate $\boldsymbol{T}(t)$. Recall our formula in definition 1.1.

$$\boldsymbol{T}(t) = \frac{1}{||\boldsymbol{r}'(t)||}\boldsymbol{r}'(t)$$

Let's find $||\boldsymbol{r}'(t)||$

$$\begin{aligned}||\boldsymbol{r}'(t)|| &= \sqrt{\frac{1}{t^2} + 0 + 1} \\ &= \sqrt{\frac{1}{t^2} + \frac{t^2}{t^2}} \\ &= \sqrt{\frac{t^2+1}{t^2}} \\ &= \frac{\sqrt{t^2+1}}{t}\end{aligned}$$

Again, we will consider the reciprocal multiplied by $\boldsymbol{r}'(t)$.

$$\boldsymbol{T}(t) = \frac{1}{||\boldsymbol{r}'(t)||}\boldsymbol{r}'(t)$$
$$\boldsymbol{T}(t) = \frac{t}{\sqrt{t^2+1}}\left\langle \frac{1}{t}, 0, 1 \right\rangle$$
$$\boldsymbol{T}(t) = \left\langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{t}{\sqrt{t^2+1}} \right\rangle$$

Sticking with our equations from definition 1.1, this algebra appears to become extraordinarily arduous. Taking the derivative componentwise of our newfound $\boldsymbol{T}(t)$ seems plenty difficult, not to mention finding the magnitude of the derivative to normalize it. It turns out that everything simplifies easily. We will do this section 6. However, let's consider an alternate route to find $\boldsymbol{N}(t)$ and see if we could avoid taking this derivative assuming we didn't know it worked out well.

**Proposition 4.1:** *$\boldsymbol{B}(t)$ and therefore $\boldsymbol{N}(t)$ can be calculated by the following formulas*

$$\boldsymbol{B}(t) = \frac{\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)}{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}$$

$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}$$

It is worth taking the time to think about where each of these formulas come from. Lets consider the first of our two new formulas. We know as a corollary from our work in section 3 that $\boldsymbol{N}$ and $\boldsymbol{T}$ form an osculating plane. More directly from our work in section 3, we know that $\boldsymbol{r}''(t)$ is always on this plane. From the definition, it is clear that $\boldsymbol{r}'(t)$ will be as well. As they are not parallel, the cross product of $\boldsymbol{r}''(t)$ and $\boldsymbol{r}'(t)$ should determine a vector that is normal to osculating plane, which–when normalized as our new equation does–is the definition of $\boldsymbol{B}$. $\boldsymbol{r}'(t)$ comes first in the formula corresponding to how $\boldsymbol{T}(t)$ comes first.

Our second formula should be intuitive from section 1, where we proved the three vectors in a Frenet frame are orthonormal. To ensure that we end up in the proper direction, let's do the following algebra to make sure we the cross product is in the proper order. We will use the following property of cross products. Suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$.

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$$

Applying this property to our known formula in definition 1.1, we can develop the following by crossing both sides with $\boldsymbol{T}$ and using the results of working with normalized vectors.

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$$
$$\boldsymbol{B} \times \boldsymbol{T} = (\boldsymbol{T} \times \boldsymbol{N}) \times \boldsymbol{T}$$
$$\boldsymbol{B} \times \boldsymbol{T} = \boldsymbol{N}(\boldsymbol{T} \cdot \boldsymbol{T}) - \boldsymbol{T}(\boldsymbol{N} \cdot \boldsymbol{T})$$
$$\boldsymbol{B} \times \boldsymbol{T} = \boldsymbol{N}(1) - \boldsymbol{T}(0)$$
$$\boldsymbol{B} \times \boldsymbol{T} = \boldsymbol{N}$$
$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}$$

We have shown where both of our new formulas come from and will now put them to use. We will begin by calculating $\boldsymbol{B}(t)$. Remember we know that

$$\boldsymbol{r}'(t) = \left\langle \frac{1}{t}, 0, 1 \right\rangle$$
$$\boldsymbol{r}''(t) = \left\langle -\frac{1}{t^2}, 0, 0 \right\rangle$$

$$\boldsymbol{B}(t) = \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{t} & 0 & 1 \\ -\frac{1}{t^2} & 0 & 0 \end{bmatrix}\right)$$

$$\boldsymbol{B}(t) = \left\langle \det\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right), -\det\left(\begin{bmatrix}\frac{1}{t} & 1\\-\frac{1}{t^2} & 0\end{bmatrix}\right), \det\left(\begin{bmatrix}\frac{1}{t} & 0\\-\frac{1}{t^2} & 0\end{bmatrix}\right)\right\rangle$$

Now we have arrived at why these alternate formulas were a useful way to approach this subsection. Both the $x$ and $z$ components of $\boldsymbol{B}(t)$ cancel out and only the $y$ component remains. By definition, $\boldsymbol{B}(t)$ is normalized, and thus if there is only a $y$ component, $\boldsymbol{B}(t)$ must equal $\langle 0, -1, 0\rangle$ for all $t$. The negative comes from solving the negative if you actually solve the determinant of the $y$ component. We could have calculated the magnitude of the cross product given in proposition 4.1, but it would unnecessary as it is simply the magnitude of a singular value.

It is worth noting that we could have deduced that $\boldsymbol{B}$ had this value simply from looking at the given equation. The $y$ component of $\boldsymbol{r}(t)$ is a constant so the osculating plane is $y = 1$. Thus, a vector orthonormal to this plane must be equal $\langle 0, 1, 0\rangle$ or $\langle 0, -1, 0\rangle$. This work will make section 6 easier.

Let's pause and remember our goal. We are using these alternate equations to find $\boldsymbol{N}(t)$. All of our hard work has left us simply having to cross $\boldsymbol{B}(t)$ with $\boldsymbol{T}(t)$ to find this vector. Recall that we found $\boldsymbol{T}(t) = \langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{t}{\sqrt{t^2+1}}\rangle$

$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}$$

$$\boldsymbol{N}(t) = \langle 0, -1, 0\rangle \times \left\langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{t}{\sqrt{t^2+1}}\right\rangle$$

$$\boldsymbol{N}(t) = \det\left(\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\0 & -1 & 0\\\frac{1}{\sqrt{t^2+1}} & 0 & \frac{t}{\sqrt{t^2+1}}\end{bmatrix}\right)$$

$$\boldsymbol{N}(t) = \left\langle \det\left(\begin{bmatrix}-1 & 0\\0 & \frac{t}{\sqrt{t^2+1}}\end{bmatrix}\right), -\det\left(\begin{bmatrix}0 & 0\\\frac{1}{\sqrt{t^2+1}} & \frac{t}{\sqrt{t^2+1}}\end{bmatrix}\right), \det\left(\begin{bmatrix}0 & -1\\\frac{1}{\sqrt{t^2+1}} & 0\end{bmatrix}\right)\right\rangle$$

$$\boldsymbol{N}(t) = \langle -\frac{t}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{t^2+1}}\rangle$$

After all our work, we can finally compare $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$. Recall that we calculated

$$\boldsymbol{r}''(t) = \langle -\frac{1}{t^2}, 0, 0\rangle$$

Comparing that to $\boldsymbol{N}(t) = \langle -\frac{t}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{t^2+1}}\rangle$, we can see that there does not exist a $\lambda$ that can directly map $\boldsymbol{N}(t)$ to $\boldsymbol{r}''(t)$. Note that $\boldsymbol{r}''(t)$ has a $z$ component of 0 and thus would not be able to achieve $\boldsymbol{N}(t)$ value in the $z$ component. Overall, it should be apparent that these two are not parallel.

We have thus achieved our goal of finding two examples, one where $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are parallel and one where they are not. These results connect very directly back to section 3 where we showed how and why $\boldsymbol{N}(t)$ and $\boldsymbol{r}''(t)$ are not parallel in the general case but may sometimes be. Furthermore, we connected to our examples in section 2, exploring them to a greater degree. We also prepared work for section 6. Overall, we have used this section to gain a greater understanding of the Frenet frame, both calculating it and working with its properties.

# 5 Introduction to Curvature

We have analyzed the Frenet frame indepth with the tools currently at our disposal. We will now consider another aspect of a vector-valued function: curvature. As the name illustrates, curvature is the measure of how vector-valued function curves. Here is the definition for curvature.

**Definition 5.1:** *If $\boldsymbol{r}'(t) \neq 0$ for all t in the domain, the curvature of a vector-valued function $\boldsymbol{r}(t)$ is defined as follows*

$$\kappa(t) = \frac{1}{||\boldsymbol{r}'(t)||}||\boldsymbol{T}'(t)||$$

In this section, we will use this formula to compute the curvature of a vector-valued function that parameterizes a circle of radius $R$. We will find the relationship between $\kappa(t)$ and $R$ in this parameterization which easily generalizes to all functions through idea of an osculating circle. An osculating circle is the approximation of a vector-valued function at a specific $t$. The curvature of this osculating circle and that of the function are the same, and thus the relationship between the curvature of a circle and that of the osculating circle will be the same.

Let us consider the following parameterization of a circle.

$$\boldsymbol{r}(t) = \langle -R\cos(t), R\sin(t), 0\rangle$$

In order to calculate the curvature, according to our formula, we need both $||\boldsymbol{r}'(t)||$ and $||\boldsymbol{T}'(t)||$. These are two values we have calculated multiple times for other more complex vector-valued functions. We will show the math briefly for these calculations. The thing to note however, is the presence of the constant $R$. Observe where it ends up.

$$\boldsymbol{r}(t) = \langle -R\cos(t), R\sin(t), 0\rangle$$
$$\boldsymbol{r}'(t) = \langle R\sin(t), R\cos(t), 0\rangle$$
$$\boldsymbol{r}''(t) = \langle R\cos(t), -R\sin(t), 0\rangle$$

We know that $\sin^2 + \cos^2 = 1$, so $|\boldsymbol{r}'(t)| = R$. This is one of the two scalars we need for our formula. Furthermore, because of this result, $\boldsymbol{T}(t)$ would normalize to be $\boldsymbol{r}'(t)$ without the $R$.

$$\boldsymbol{T}(t) = \frac{1}{R}\langle R\sin(t), R\cos(t), 0\rangle$$
$$\boldsymbol{T}(t) = \langle \sin(t), \cos(t), 0\rangle$$
$$\boldsymbol{T}'(t) = \langle \cos(t), -\sin(t), 0\rangle$$

We can clearly see from this result that the magnitude of $\boldsymbol{T}'(t) = 1$. Let's now evaluate using our formula for curvature.

$$\kappa(t) = \frac{1}{||\boldsymbol{r}'(t)||}||\boldsymbol{T}'(t)||$$

$$\kappa(t) = \frac{1}{R}(1)$$

$$\kappa(t) = \frac{1}{R}$$

Thus, we have found that curvature is inversely proportional to the radius of an osculating circle. We already described how it will generalize to all function. Normally, $\kappa(t)$ is not constant. It will change with the changing radius of the osculating circle that approximates a vector-valued function it describes. We will consider an example of this in the following sections.

# 6 Finding an Osculating Circle

In this section, we will determine an osculating circle for a vector-valued function at a specific point. This will put into practice our formula for curvature and generalization of our idea in section 5. We will take this relationship between the radius of an osculating circle and curvature at a point and apply it to a specific example.

Moreover, we will use our example from section 4, subsection ii. Our goal will be to find the osculating circle to the below function at $t = 1$.

$$\boldsymbol{r}(t) = \langle \ln(t), 1, t\rangle$$

Recall our formula

$$\kappa(t) = \frac{1}{||\boldsymbol{r}'(t)||}||\boldsymbol{T}'(t)||$$

We will use our knowledge of how to calculate the many vectors and magnitudes associated with Frenet frame to find our two needed scalars in this formula. Remember, we have already done a great deal of work with this particular vector-valued function and will use that to our advantage. Thus, let's attempt to find $||\boldsymbol{r}'(t)||$ and $||\boldsymbol{T}'(t)||$

in order to find curvature. We already know that curvature is inversely proportional to the radius of our osculating circle. Thus, this will be the crux our work in this section. We will go faster in our calculations as we have now done this algebra many times over.

Let's copy down what we already know:

$$\begin{aligned}
\boldsymbol{r}'(t) &= \langle \frac{1}{t}, 0, 1 \rangle \\
\boldsymbol{r}''(t) &= \langle -\frac{1}{t^2}, 0, 0 \rangle \\
\boldsymbol{T}(t) &= \langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{t}{\sqrt{t^2+1}} \rangle \\
||\boldsymbol{r}'(t)|| &= \frac{\sqrt{t^2+1}}{t}
\end{aligned}$$

We explored an alternate route to avoid taking the derivative of $\boldsymbol{T}(t)$ in section 4. This made it both easier for us in the problem and allowed us to bring our understanding of Frenet frames further. Unfortunately, we must do the algebra here. We will take the derivative of $\boldsymbol{T}(t)$ componentwise and then find the magnitude.

$$\begin{aligned}
\boldsymbol{T}(t) &= \langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{t}{\sqrt{t^2+1}} \rangle \\
\boldsymbol{T}'(t) &= \langle \frac{-t}{(t^2+1)^{\frac{3}{2}}}, 0, \frac{1}{(t^2+1)^{\frac{3}{2}}} \rangle
\end{aligned}$$

Now, let's calculate the magnitude of $\boldsymbol{T}'(t)$. It works out nicely.

$$\begin{aligned}
||\boldsymbol{T}'(t)|| &= \sqrt{\left(\frac{-t}{(t^2+1)^{\frac{3}{2}}}\right)^2 + \left(\frac{1}{(t^2+1)^{\frac{3}{2}}}\right)^2} \\
&= \sqrt{\frac{t^2}{(t^2+1)^3} + \frac{1}{(t^2+1)^3}} \\
&= \sqrt{\frac{t^2+1}{(t^2+1)^3}} \\
&= \sqrt{\frac{1}{(t^2+1)^2}} \\
&= \frac{1}{t^2+1}
\end{aligned}$$

It is time to plug in our value of $t = 1$. The algebra certainly could have been more arduous than it was. Finding the magnitude allowed for terms to cancel nicely. Recall that we are looking for our two scalar terms in the formula for curvature. Thus, we need to evaluate the following two terms at $t = 1$.

$$||\boldsymbol{r}'(t)|| = \frac{\sqrt{t^2+1}}{t} \text{ and } ||\boldsymbol{T}'(t)|| = \frac{1}{t^2+1}$$

We see that

$$||\boldsymbol{r}'(1)|| = \sqrt{2} \text{ and } ||\boldsymbol{T}'(1)|| = \frac{1}{2}$$

Finally, let's apply the formula.

$$\kappa(t) = \frac{1}{||\boldsymbol{r}'(t)||}||\boldsymbol{T}'(t)||$$
$$\kappa(1) = \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$\kappa(1) = \frac{1}{2\sqrt{2}}$$

Okay, we have now calculated our curvature. Recall that our goal is to determine the osculating circle at $t = 1$. Thus, we need to consider the three things that define a circle in $\mathbb{R}^3$.

1. The radius of the circle
2. The center of the circle
3. The plane containing the circle

First, let's focus on the radius. Remember from section 5 that $\kappa(t) = \frac{1}{R}$. Recall how this relationship generalizes to osculating circle as it is part of the definition of the osculating circle. We can easily find our radius at $t = 1$.

$$\kappa(1) = \frac{1}{R}$$
$$\frac{1}{2\sqrt{2}} = \frac{1}{R}$$
$$R = 2\sqrt{2}$$

Continuing in the order listed, let's attempt to find the center of the circle. This connects back all the way to section 1 and our understanding of how $\boldsymbol{N}$ is orthonormal to $\boldsymbol{T}$. We know that $\boldsymbol{T}$ will be tangent to our osculating circle by definition. $\boldsymbol{T}$ and the osculating circle are both related to approximating the curve and thus will be $\boldsymbol{T}$ will be tangent to the circle at $t = 1$. We can use this idea because it implies that $\boldsymbol{N}$ at $t = 1$ will point directly inwards toward the center of the circle. Thus, starting at the value of $\boldsymbol{r}(1)$, we can go $R$ units in the direction of $\boldsymbol{N}(1)$ to find our center. Recall that $\boldsymbol{N}(1)$ is normalized. We will determine the center as follows

In section 4, we calculated that $\boldsymbol{N}(t) = \langle -\frac{t}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{t^2+1}}$. This leaves $\boldsymbol{N}(1) = \langle -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\rangle$.

$$\boldsymbol{r}(1) + R\boldsymbol{N}(1)$$
$$\langle 0, 1, 1\rangle + 2\sqrt{2}\langle -\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\rangle$$
$$\langle 0, 1, 1\rangle + \langle -2, 0, 2\rangle$$
$$\langle -2, 1, 3\rangle$$

Thus, our radius is $2\sqrt{2}$ and our center is $\langle -2, 1, 3\rangle$. For our last data point, we need to determine the plane the circle lies in. This will just be the plane defined by the vector normal to the osculating circle which is by definition $\boldsymbol{B}(1)$. This should be clear as a product of our result from section 3. From our work in section 4, we know that it is normal to $\langle 0, \lambda, 0\rangle$ where $\lambda \in \mathbb{R}$. Put another way, the circle will lie in the plane $y = 1$. Similar to our closing analysis in section 4, this should be clear from the fact that the $y$-component of our function is constant.

Thus, we have successfully determined our circle. It is normally difficult to determine a parameterization of a circle in $\mathbb{R}^3$, but it is certainly possible. However, because it lies on $y = 1$, we can relatively easily put our information together and parameterize the circle like so. Suppose the circle is a function $\boldsymbol{C}(t) : \mathbb{R} \mapsto \mathbb{R}^3$.

$$\boldsymbol{C}(t) = \langle 2\sqrt{2}\cos(t) - 2, 1, 2\sqrt{2}\sin(t) + 3\rangle$$

We have successfully applied our knowledge of the Frenet frame and of curvature to a specific example, demonstrating our ability to put all of our work so far together.

# 7 An Alternate Formula for Curvature

We have worked with Frenet frames in a number of ways. It should be clear now that many of the vectors and magnitudes in the Frenet frame have many intertwined relationships with one another. The last new relationship we will consider is an alternative formula for curvature. In the last two sections, we have worked indepth with curvature. Now, we will make a claim about a new formula for curvature and prove it using many branches of the knowledge we have developed so far.

**Proof:**

We claim and want to show that the following is true

$$\kappa(t) = \frac{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3}$$

We know that

$$\kappa(t) = \frac{1}{||\boldsymbol{r}'(t)||}||\boldsymbol{T}'(t)||$$

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{||\boldsymbol{r}'(t)||}$$

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{||\boldsymbol{T}'(t)||}$$

Let us consider what we know

$$\kappa(t) = \frac{||\boldsymbol{T}'(t)||}{||\boldsymbol{r}'(t)||}$$

We can then multiply by 1 twice

$$\kappa(t) = \frac{||\boldsymbol{T}'(t)||||\boldsymbol{T}(t)|| \sin\left(\frac{\pi}{2}\right)}{||\boldsymbol{r}'(t)||}$$

Recognize definition 1.3–the cross product formula–in the numerator

$$\kappa(t) = \frac{||\boldsymbol{T}(t) \times \boldsymbol{T}'(t)||}{||\boldsymbol{r}'(t)||}$$

$$\kappa(t) = \frac{||\boldsymbol{r}'(t)||^2 \left(||\boldsymbol{T}(t) \times \boldsymbol{T}'(t)||\right)}{||\boldsymbol{r}'(t)||^3}$$

Let's pause and consider what we know. By rearranging the primary Frenet frame formulas listed above, we have the following.

$$\boldsymbol{r}'(t) = \boldsymbol{T}(t)||\boldsymbol{r}'(t)||$$

Then, if we differentiate each side with respect to $t$ using the product rule, we have

$$\boldsymbol{r}''(t) = \boldsymbol{T}(t)||\boldsymbol{r}'(t)||' + \boldsymbol{T}'(t)||\boldsymbol{r}'(t)||$$

Let us do a side proof to build on what we have. We claim that

$$||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|| = ||\boldsymbol{r}'(t)||^2 \left(||\boldsymbol{T}(t) \times \boldsymbol{T}'(t)||\right)$$

Consider

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$$

We can substitute in our modified formula for $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \boldsymbol{T}(t)||\boldsymbol{r}'(t)|| \times \left(\boldsymbol{T}(t)||\boldsymbol{r}'(t)||' + \boldsymbol{T}'(t)||\boldsymbol{r}'(t)||\right)$$

We can simplify by distributing the cross product and factoring out our scalars. Remember, the cross product of a vector with itself is the zero vector.

$$\begin{aligned} \boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= \boldsymbol{T}(t)||\boldsymbol{r}'(t)|| \times (\boldsymbol{T}(t)||\boldsymbol{r}'(t)||') + \boldsymbol{T}(t)||\boldsymbol{r}'(t)|| \times (\boldsymbol{T}'(t)||\boldsymbol{r}'(t)||) \\ &= \boldsymbol{T}(t)||\boldsymbol{r}'(t)|| \times (\boldsymbol{T}'(t)||\boldsymbol{r}'(t)||) \\ &= ||\boldsymbol{r}'(t)||^2 \left(\boldsymbol{T}(t) \times \boldsymbol{T}'(t)\right) \end{aligned}$$

Let's substitute this result back into our main proof. We left off with

$$\kappa(t) = \frac{||\boldsymbol{r}'(t)||^2 \left(||\boldsymbol{T}(t) \times \boldsymbol{T}'(t)||\right)}{||\boldsymbol{r}'(t)||^3}$$

$$\kappa(t) = \frac{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3}$$

□

Thus, we haven our alternate formula for curvature. It uses a variety of the relationships between vectors in the Frenet frame that we have studied. It builds off of of our understanding of curvature from sections 5 and 6.

## 8 Summary

In this report, we developed a nuanced understanding of the Frenet frame, curvature and how vector-valued functions from $\mathbb{R} \mapsto \mathbb{R}^3$ interact with these two concepts. We analyzed the properties of the Frenet frame, gave some examples, and then saw how it applies to curvature. In writing this report, I took ideas from office hours both with Dr. Wong and with the LAs. I collaborated with Jack Hambidge and with Lucas Shardt. I also watched a YouTube video (https://youtu.be/LwOorOh6Wt4) for some initial context about the problem. I found an alternative formula for $\boldsymbol{B}$ here: https://math.etsu.edu/multicalc/prealpha/Chap1/Chap1-8/part4.htm. I took definitions and theorems from the lecture notes and from the challenge report assignment.