Challenge Problem Report 2

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November 16, 2023

This challenge report explores limits, specifically using polar coordinates in $\mathbb{R}^2$. We show that mapping to polar is not a linear map. We show that there are many ways to represent a point in polar and graph a polar equation. Then, we transition into limits where we show how we can use polar as a tool through a specific application of the squeeze theorem to calculate and determine if limits exist.

1 Linearity of Polar Coordinates

We will begin by showing that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is not linear. This map is defined by:

$$(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$$

In order to determine linearity, we must look at a definition of a linear map.

Definition 1.1: A map $g : V \mapsto W$ is a linear map if and only if the following two conditions are true:

1. For arbitrary vectors $\vec{u}, \vec{v} \in V, g(\vec{u} + \vec{v}) = g(\vec{u}) + g(\vec{v})$
2. For an arbitrary $\vec{u} \in V$ and any $\lambda \in \mathbb{R}$, $g(\lambda\vec{u}) = \lambda g(\vec{u})$

Proving that our map f does not satisfy either of these conditions every time is sufficient to prove that f is not linear. Thus, we will look at the first condition, the additive aspect of linear maps and see how it relates to our map f.

Suppose $\vec{a}, \vec{b} \in \mathbb{R}^2$ such that $\vec{a} = \langle r_a, \theta_a \rangle$ and $\vec{b} = \langle r_b, \theta_b \rangle$

Consider:

$$f(\vec{a} + \vec{b})$$
$$= f((r_a, \theta_a) + (r_b, \theta_b))$$
$$= f((r_a, \theta_a) + (r_b, \theta_b))$$

By the component-wise definition of vector addition in $\mathbb{R}^2$

$$= ((r_a + r_b)\cos(\theta_a + \theta_b), (r_a + r_b)\sin(\theta_a + \theta_b))$$

By the definition of f

Comparing this with the value of $f(\vec{a}) + f(\vec{b})$ yields

$$= (r_a \cos\theta_a, r_a \sin\theta_a) + (r_b \cos\theta_b, r_b \sin\theta_b)$$
$$= (r_a \cos\theta_a + r_b \cos\theta_b, r_a \sin\theta_a + r_b \sin\theta_b)$$

Observe:

$$((r_a + r_b)\cos(\theta_a + \theta_b), (r_a + r_b)\sin(\theta_a + \theta_b)) \neq (r_a \cos\theta_a + r_b \cos\theta_b, r_a \sin\theta_a + r_b \sin\theta_b)$$

With some values of $\vec{a}, \vec{b}$ this may hold. However, it clearly does not hold for all values. For instance, a counter example could be:

$$\vec{a} = \langle 1, \frac{\pi}{3} \rangle, \vec{b} = \langle 5, \frac{\pi}{4} \rangle$$

$$\Big((1+5)\cos(\frac{\pi}{3}+\frac{\pi}{4}),(1+5)\sin(\frac{\pi}{3}+\frac{\pi}{4})\Big) \neq \Big(1\cos(\frac{\pi}{3})+5\cos(\frac{\pi}{4})),1\sin(\frac{\pi}{3})+5\sin(\frac{\pi}{4})\Big)$$

$$(-1.5529, 5.7955) \neq (4.0355, 4.40155)$$

Because we are trying to disprove, we only need one counterexample. However, it is clear that most often

$$((r_a + r_b)\cos(\theta_a + \theta_b), (r_a + r_b)\sin(\theta_a + \theta_b)) \neq (r_a \cos\theta_a + r_b \cos\theta_b, r_a \sin\theta_a + r_b \sin\theta_b)$$

$$f(\vec{a} + \vec{b}) \neq f(\vec{a}) + f(\vec{b})$$

Thus, f is not a linear map. $\square$

2 Infinite Representations of Euclidean Coordinates as Polar Coordinates

We will now look at how Euclidean coordinates can be expressed as polar coordinates. We will show how they can be expressed in an infinite number of ways, showing there are an infinite number of pairs of (r, θ) that correspond to a single (a, b).

Definition 2.1: The arbitrary rectangular point (x, y) corresponds to the point (r, θ) in polar form by the following:

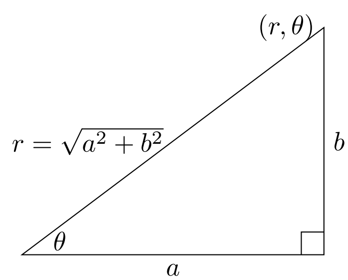
$$r = \sqrt{x^2 + y^2} \text{ and } \tan(\theta) = \frac{y}{x} \text{ if } x \neq 0$$

Definition 2.2: The arbitrary polar point (r, θ) corresponds to the point (x, y) in rectangular form by the following:

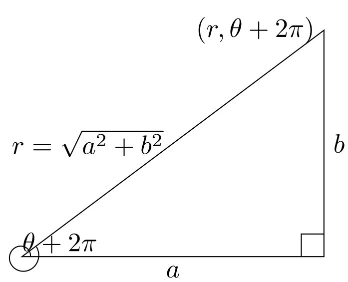
$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

The above definitions were modified from the challenge problem report definitions 1.1 and 1.2 respectively

Consider the following diagram:

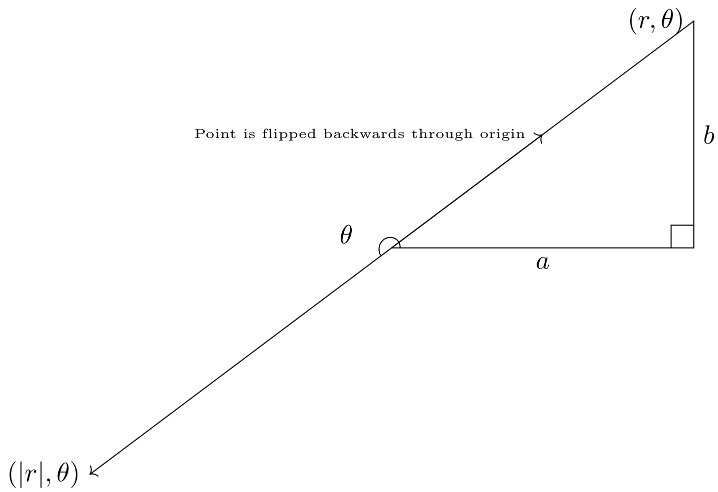


Let's use this diagram to give a visual intuition for why there exist an infinite number of polar representations of a single rectangular point. Note how the same point can be represented by adding a factor of 2π.



Visually, it should be clear that this process of adding a factor of 2π will alone produce an infinite number of representations of a rectangular coordinate. Imagine the angle progressing clockwise instead. This would occur if factors of 2π are subtracted from θ, which, again, can be done an infinite number of times.

Now, let's consider how a negative r can be corrected of θ by π. In the following picture, suppose r is negative and θ is a value greater than π.



Now, let's show this algebraically:

Using definition 2.1, we see that

$$r = \sqrt{a^2 + b^2} \text{ and } \tan(\theta) = \frac{b}{a} \text{ if } a \neq 0$$

We can thus deduce that:

$$(r, \theta) = (-r, \theta - \pi)$$

Furthermore, we know by the periodic nature of a polar angle θ:

$$(r, \theta) = (r, \theta + 2\pi n) \text{ where } n \in \mathbb{Z}$$

The above statement produces an infinite number of potential representations of a rectangular coordinates, showing both algebraically and geometrically how there are multiple ways to represent a singular Euclidean coordinate in polar coordinates.

For good measure, we will consider an example. Suppose we want to convert $(3, 4)$ into polar coordinates. We will do the following:

$$r = \sqrt{3^2 + 4^2} \text{ and } \tan(\theta) = \frac{4}{3}$$

We can say that:

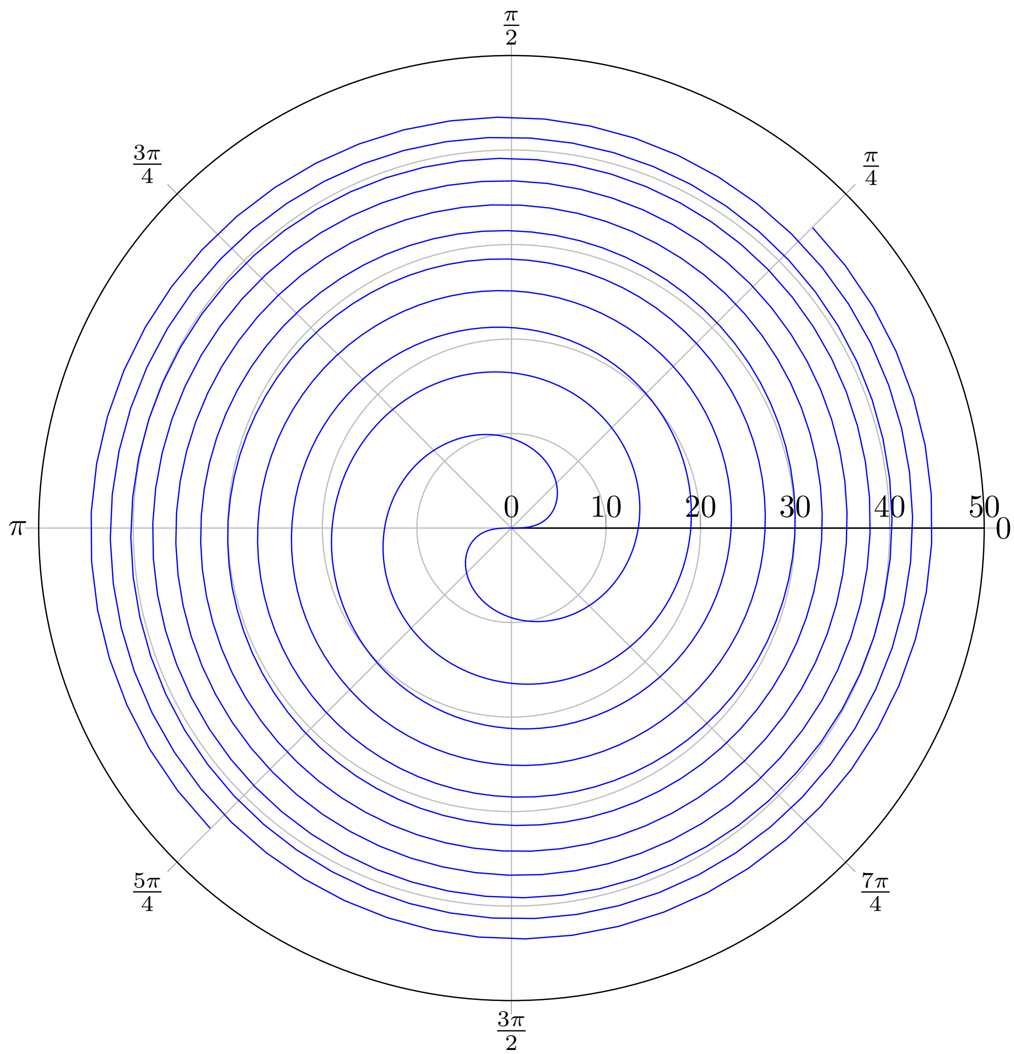
$$(3, 4) \longrightarrow (5, 0.927 + 2\pi n) \text{ and } (-5, 4.069 + 2\pi n)$$

This covers all the ways this point can be expressed.

3 Graph of a Polar Equation

We will now graph the equation $\theta = r^2$. It is worth recognizing that this graph is slightly more unique as θ is written as a function of r. More often than not, r is the dependent variable and θ is the independent variable in polar equations. Because of this distinction, to make the actual graph below, the code uses the function split into two parts as a function of θ instead of one of r.

$r = \sqrt{\theta}$ and $r = -\sqrt{\theta}$



Note: the code to create the above graph $\theta = r^2$ was modified from the code in the challenge report itself.
Here is a table of values to consider that we could have used to begin to create our graph:

r	θ
1	1
-1	1
2	4
-2	4
π	π^2
$-\pi$	π^2

From this table, it is easy to see how r will have both a positive and negative value for each value of θ

4 Limits Using Polar Coordinates

We have the following theorem. **Theorem 4.1:** Limits with polar coordinates (This is taken nearly directly from Theorem 2.2 of the challenge report.)

Suppose $f(x, y) : \mathbb{R}^2 \mapsto \mathbb{R}$ is a function of two variables that can be expressed in polar coordinates as $g(r, \theta) := f(r\cos(\theta), r\sin(\theta))$. Then,

$$\lim_{(x,y)\to(0,0)} f(x, y) = L$$

if and only if there exists $\delta > 0$ and a function $h : \mathbb{R} \mapsto \mathbb{R}$ such that both

- If $0 < r < \delta$, then $|g(r, \theta) - L| \leq h(r)$ for all θ
- $\lim_{r \to 0} h(r) = 0$

In the section, we will come up with an analog for this theorem such that we can use for limits when they approach any (a, b) as opposed to just $(0, 0)$.

Theorem 4.1 geometrically states that has a limit of 0 when x, y approach 0 because of the following. We know that

$$r = \sqrt{x^2 + y^2}$$

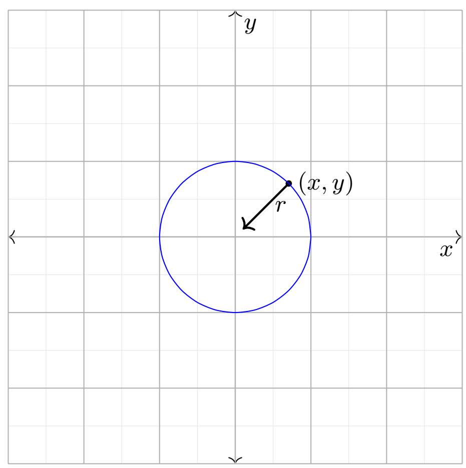
and thus

$$r^2 = x^2 + y^2$$

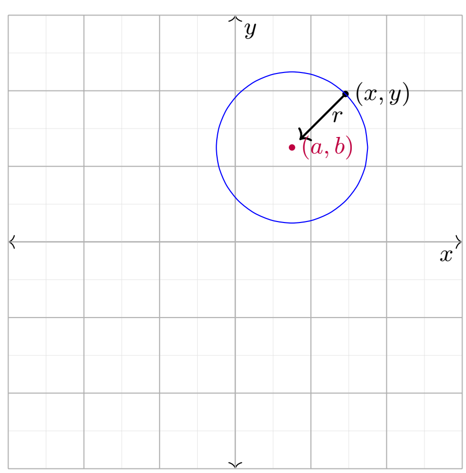
Therefore, we can say that as $\lim_{(x,y) \to (0,0)}$, $f(x, y)$ stays inside the following:

$$\{(x, y) | x^2 + y^2 \leq r^2\}$$

.



Now, what we want to do is shift our theorem so that we can see the behavior of a function in as inputs approach any a, b. This would correlate to the following change:



Now, we will state the analogous theorem taking into account the above shift:

Theorem 4.2: Analogous theorem to Theorem 4.1

Suppose $f(x, y) : \mathbb{R}^2 \mapsto \mathbb{R}$ is a function of two variables that can be expressed in polar coordinates as $g(r, \theta) := f(r \cos(\theta) + a, r \sin(\theta)) + b)$. Then,

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if and only if there exists $\delta > 0$ and a function $h : \mathbb{R} \mapsto \mathbb{R}$ such that both

- If $0 < r < \delta$, then $|g(r, \theta) - L| \leq h(r)$ for all θ

- $\lim_{r\to 0} h(r) = 0$

In order to prove Theorem 4.2, we will reduce to proving Theorem 4.1 as follows.
We will define the following:

$$\tilde{f}(x,y) = f(x+a, x+b)$$

and

$$u = x - a \text{ and } v = y - b$$

Consider:

$$\begin{aligned} L &= \lim_{(x,y)\to(a,b)} f(x,y) \\ &= \lim_{(x-a,y-b)\to(0,0)} f(x,y) \\ &= \lim_{(u,v)\to(0,0)} f(x,y) \\ &= \lim_{(u,v)\to(0,0)} f(u+a, v+b) \\ &= \lim_{(u,v)\to(0,0)} \tilde{f}(u,v) \end{aligned}$$

Thus, we can clearly see that proving Theorem 4.1 means that Theorem 4.2 is also true. This follows both from the algebra and from the geometric picture which essentially just shows a translation of the theorem, a kind of moving of the origin.

As such, we will now prove Theorem 4.1. Because this theorem is a biconditional statement, we should normally prove the theorem in both ways. This would mean assuming that the limit of $f(x,y) = L$ and deriving the two conditions from that, and it would mean starting with the two conditions and getting back to that. However, for our purposes, the useful way is to start with the conditions and show that together they imply that $\lim_{(x,y)\to(0,0)} f(x,y) = L$. This is useful because we generally use the conditions to determine the limit when using this theorem.

First, we must consider the very fundamental $\delta - \epsilon$ definition of a limit.

Definition 4.1: $\delta - \epsilon$ limit definition of a multivariable function.

A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ has the limit L at a certain value a if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\vec{x} \in \mathbb{R}^n$,

$$0 < ||\vec{x} - \vec{a}|| < \delta \text{ implies that } |f(\vec{x}) - L| < \epsilon$$

Thus, in order to prove that $\lim_{(x,y)\to(0,0)} f(x,y) = L$, we want the above definition to be true.

Proof:

Suppose $\exists h : \mathbb{R} \to \mathbb{R}$ and $\delta > 0$ that satisfies the above conditions. Consider the following. We will convert into rectangular coordinates and show how this reduces to our definition.

$$\begin{gathered} \text{if } 0 < r < \delta, \text{ then } |g(r,\theta) - L| < h(r) \\ \leftrightarrow \text{if } 0 < \sqrt{x^2+y^2} < \delta, \text{ then } |f(x,y) - L| \leq h(r) \\ \leftrightarrow \text{if } 0 < || < x, y > || < \delta, \text{ then } |f(x,y) - L| \leq h(r) \\ \leftrightarrow \text{if } 0 < || < x, y > - < 0, 0 > || < \delta, \text{ then } |f(x,y) - L| \leq h(r) \\ \leftrightarrow \text{if } 0 < ||\vec{x} - \vec{0}|| < \delta, \text{ then } |f(x,y) - L| \leq h(r) \end{gathered}$$

As $\lim_{r\to 0} h(r) = 0$, we can say that for all $\epsilon_h > 0, \exists \delta_h > 0$ such that

$$0 < |r - 0| < \delta_h \text{ implies that } |h(r) - 0| < \epsilon_h$$

Thus, if we set $\epsilon_h = \epsilon$ and since we have already proven the condition immediately above regarding δ_h, we can work with the following

$$\begin{gathered} |h(r) - 0| < \epsilon \\ |h(r)| < \epsilon \end{gathered}$$

Now substituting into our earlier work:

$$\leftrightarrow \text{if } 0 < ||\vec{x} - \vec{0}|| < \delta, \text{ then } |f(x,y) - L| \leq h(r)$$

$$\leftrightarrow \text{if } 0 < ||\vec{x} - \vec{0}|| < \delta, \text{ then } |f(x,y) - L| < \epsilon$$

We have thus proven that $\lim_{(x,y)\to(0,0)}$ exists and has a value of L if $\lim_{r\to 0} h(r) = 0$ and if $0 < r < \delta$, then $|g(r,\theta) - L| \leq h(r)$ for all θ. Therefore, by definition 4.1, we have proven the useful direction of Theorem 4.1, proving Theorem 4.2 in the process. $\square$

5 Showing Limits Do Not Exist with Polar Coordinates

In this section, we will attempt to prove that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

In order to consider limits using polar coordinates, we need to consider another theorem.

Theorem 5.1: Squeeze Theorem

Suppose $f(\vec{x}, g(\vec{x}), h(\vec{x})$ are multivariable functions from $\mathbb{R}^n \mapsto \mathbb{R}$ such that:

$$\lim_{\vec{x}\to\vec{P}} f(\vec{x}) = L = \lim_{\vec{x}\to\vec{P}} h(\vec{x})$$

***If** there exists $\delta > 0$ such that for all $\vec{x} \in B_\delta(\vec{P}) - \{\vec{P}\}$ the following is true*

$$f(\vec{x}) \leq g(\vec{x}) \leq h(\vec{x})$$

Then $\lim_{\vec{x}\to\vec{P}} g(\vec{x}) = L$

In general terms, the squeeze theorem states that if a function is squeezed between two functions and both functions approach the same value L at a certain point, the limit of the function being squeezed is also L.

Looking again at our limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

We will convert it into polar form using definition 2.1.

$$\lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$\leftrightarrow \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2}$$
$$\leftrightarrow \lim_{r\to 0} \cos^2(\theta)$$
$$\leftrightarrow \cos^2(\theta)$$

We see that this limit now depends on θ. We have a corollary to theorem 4.2 (taken directly from the Challenge Problem Report) which is the following

Corollary 5.1: If $\lim_{r\to 0} g(r,\theta)$ depends on θ, then the value of the limit will differ for different straight line paths. Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Lets discuss the above corollary. For a limit to exist, it must have a singular value. We have proven the uniqueness of limits before. Thus, it should suffice to evaluate the value of the limit at distinct values of θ and observe how they are different to prove the limit doesn't exist.

Consider $\theta = 0$ and $\theta = \frac{\pi}{4}$ of $\cos^2(\theta)$:

$$= \cos^2(0) = 1^2 = 1 \qquad\qquad = \cos^2\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

Because $\frac{1}{2} \neq 1$, we know that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

does not exist. $\square$

6 Subtleties of Limits with Polar Coordinates

The following section will attempt to prove both that the limit of the function below is 0 and that it doesn't exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

More specifically, it will prove that when f is converted into polar coordinates, $\lim_{r\to 0} g(r,\theta) = 0$ while proving that f in polar form does not have a limit. We will then relate this to Theorem 4.1.

i. Proving $\lim_{r\to 0} g(r,\theta) = 0$

We need to consider the following properties of limits. These properties were taken from the lecture notes. We will employ most of these properties in the following algebra. Suppose $f, g : \mathbb{R}^n \mapsto \mathbb{R}$ and the limits as both inputs approach $\vec{P}$ exist.

a. Sum Law:

$$\lim_{\vec{x}\to\vec{P}} (f(\vec{x}) + g(\vec{x})) = \lim_{\vec{x}\to\vec{P}} f(\vec{x}) + \lim_{\vec{x}\to\vec{P}} g(\vec{x})$$

b. Scalar Multiple Law:

$$\lim_{\vec{x}\to\vec{P}} \lambda(f(\vec{x})) = \lambda \lim_{\vec{x}\to\vec{P}} f(\vec{x})$$

c. Product Law:

$$\lim_{\vec{x}\to\vec{P}} (f(\vec{x})g(\vec{x})) = \left(\lim_{\vec{x}\to\vec{P}} f(\vec{x})\right)\left(\lim_{\vec{x}\to\vec{P}} g(\vec{x})\right)$$

d. Quotient Law: If $\lim_{\vec{x}\to\vec{P}} g(\vec{x}) \neq 0$,

$$\lim_{\vec{x}\to\vec{P}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x}\to\vec{P}} f(\vec{x})}{\lim_{\vec{x}\to\vec{P}} g(\vec{x})}$$

Consider:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} &= \lim_{r\to 0} \frac{r\cos(\theta)r^2\sin^2(\theta)}{r^2\cos^2(\theta) + r^4\sin^4(\theta)} \\
&= \lim_{r\to 0} \frac{r\cos(\theta)\sin^2(\theta)}{\cos^2(\theta) + r^2\sin^4(\theta)} \\
&\textit{we are now taking the case where the denominator} \neq 0 \\
&= \frac{\lim_{r\to 0} r\cos(\theta)\sin^2(\theta)}{\lim_{r\to 0}\left(\cos^2(\theta) + r^2\sin^4(\theta)\right)} \\
&= \frac{\lim_{r\to 0} r \times \lim_{r\to 0}\cos(\theta)\sin^2(\theta)}{\lim_{r\to 0}(\cos^2(\theta)) + \lim_{r\to 0}\left(r^2\sin^4(\theta)\right)} \\
&\textit{Because } \sin(\theta) \textit{ and } \cos(\theta) \textit{ range between 1 and -1,} \\
&\textit{the squeeze theorem can be applied.} \\
&= \frac{0}{\cos^2(\theta)}
\end{aligned}$$

The above will have a limit of 0 whenever $\cos^2(\theta) \neq 0$. Now, since our goal is to show that the limit is 0, we must show that when $\cos^2(\theta) = 0$, the limit is still 0. We know that $\cos^2(\theta) = 0$ when $\theta = \frac{\pi}{2} + \pi n$. Let us jump back to an earlier step:

$$\lim_{r\to 0} \frac{r\cos(\theta)\sin^2(\theta)}{\cos^2(\theta) + r^2\sin^4(\theta)}$$

plugging in 0 for all values of $\cos(\theta)$ produces

$$\lim_{r\to 0} \frac{0}{r^2}$$

From single variable calculus, we know this limit is 0. Thus, we have shown based on theorem 4.1, $\lim_{r\to 0} g(r,\theta) = 0$.

ii. Proving $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ **does not exist**

We will first consider the following definition and theorem (both were taken from the lecture notes; def. 2.3.9 and theorem 2.3.10):

Definition 6.1: *Let $X \subset \mathbb{R}^n$. A point $\vec{p} \in \mathbb{R}^n$ is a limit point of X if there is a sequence $\{\vec{a}_n\}$ contained inside X such that $\{\vec{a}_n\}$ converges to $\vec{p}$*

Theorem 6.1: *Let $X \subset \mathbb{R}^n$, $f : X \mapsto \mathbb{R}^m$ be a function and $\vec{a}$ be a limit point of X. Then, the below two statements are equal:*

1. $\lim_{\vec{x}\to\vec{a}} f(x) = \vec{b}$

2. *All sequences* $\{\vec{a}_n\}$ *converging to* $\vec{a}$ *where* $\vec{a}_n \neq \vec{a}$*, the sequence* $\{f(\vec{a}_n)\}$ converges to $\vec{b}$

What this theorem and the definition supplementing it tell us is that for the limit to exist, it must be equivalent across any and all approaches to said limit. Thus, if we find two distinct paths $\vec{r_1}(t)$ and $\vec{r_2}(t)$ such that they do not both produce the same limit, then the limit does not exist.

Consider the following two approaches to $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

$$\vec{r_1}(t) = \langle 0, t\rangle \text{ and } \vec{r_2}(t) = \langle t^2, t\rangle$$

Consider $\vec{r_1}(t)$:

$$\frac{0 \times t^2}{0 + t^4} = 0$$

Consider $\vec{r_2}(t)$:

$$\frac{t^2 \times t^2}{t^4 + t^4} = \frac{1}{2}$$

Because $\frac{1}{2} \neq 1$, the $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

iii. Relate to Theorem 4.1

We will now attempt to reconcile our results. The root of the difference between the results comes from losses during our simplification down from a two variable system to a one variable system where we had to fix our analysis of θ.

The issue arises with a specific element of Theorem 4.1. The conditions of Theorem 4.1 require that If $0 < r < \delta$, then $|g(r,\theta) - L| \leq h(r)$ **for all** θ

The key is for all θ. This means that the theorem must hold true even for when $\theta = f(r)$ (not to be confused with our primary multivariable function f). We will now show a path such that this limit does not equal 0. Once we do that, the seemingly contradictory results explain themselves. Our segmented analysis does not allow θ to take curved paths. While we still cover the entire plane in our single variable case, we do not cover this $\theta = f(r)$ case.

Consider the following $\theta = f(r)$:

$$\theta = \cos^{-1}(r)$$

We will again go back to the second step in section i and then substitute in this path.

$$\begin{gathered}
\lim_{r\to 0} \frac{r\cos(\theta)\sin^2(\theta)}{\cos^2(\theta) + r^2\sin^4(\theta)} \\
\leftrightarrow \lim_{r\to 0} \frac{r\cos(f(r))\sin^2(f(r))}{\cos^2(f(r)) + r^2\sin^4(f(r))} \\
\leftrightarrow \lim_{r\to 0} \frac{r\cos(\cos^{-1}(r))\sin^2(\cos^{-1}(r))}{\cos^2(\cos^{-1}(r)) + r^2\sin^4(\cos^{-1}(r))} \\
\leftrightarrow \lim_{r\to 0} \frac{r^2\sin^2(\cos^{-1}(r))}{r^2 + r^2\sin^4(\cos^{-1}(r))} \\
\leftrightarrow \lim_{r\to 0} \frac{\sin^2(\cos^{-1}(r))}{1 + \sin^4(\cos^{-1}(r))} \\
\text{by knowledge of single variable limits} \\
\leftrightarrow \frac{\sin^2(\cos^{-1}(0))}{1 + \sin^4(\cos^{-1}(0))} \\
\leftrightarrow \frac{\sin^2(\frac{\pi}{2})}{1 + \sin^4(\frac{\pi}{2})} \\
\leftrightarrow \frac{1}{1+1} \\
\leftrightarrow \frac{1}{2}
\end{gathered}$$

Thus, we have found a path $\theta = f(r)$ that does not have a limit of 0. This was the primary issue. By our analysis in part i, we made $\theta \in \mathbb{R}$, some fixed value. This was acceptable for our limit of $\lim_{r\to 0} g(r,\theta)$. The limit of this is definitively 0. Say $r = t$ and $\theta = f(t)$. We were not taking the limit as $\lim_{t\to 0} g(t, f(t))$; thus, we did not consider anything other than straight line paths. We ignored when θ would pass through $\frac{\pi}{2}$ or some multiple of it. Thus, segmented analysis would very clearly break down because it does not cover when θ is in both categories.

We can also see how $h(r)$ and thus the squeeze theorem that theorem 4.1 is based on breaks down. $|g(r,\theta) - L| \leq h(r)$ should be true given the conditions. However, we see that:

$$\lim_{r \to 0} \frac{r\cos(\theta)\sin^2(\theta)}{\cos^2(\theta) + r^2\sin^4(\theta)} \le h(r)$$

contradicting $\lim_{r\to 0} h(r) = 0$. This provides yet another example of the breakdown of Theorem 4.1 when θ cannot be completely reduced out of the equation. This is the idea of our Corollary 5.1. $h(r)$ is a collection of circular level curves that compress into a singular point. However, this fails with our path $\theta = f(r)$.

Summary:

In this report we proved and discussed theorems and examples involving polar coordinates and limits. We looked at nuances of how polar coordinates can and cannot be used. I collaborated with many people on this report. I learned a lot from office hours with both LAs and with Professor Wong. I took ideas about question four, five, and six from these office hours. Furthermore, I collaborated somewhat with about five students in the class: Jack, Bogdan, Hero, Ishan, and Lucas. Additionally, Saf reviewed my report and gave some feedback on question 6.