Challenge Problem Report 1

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This challenge report explores linear maps in $\mathbb{R}^2$, focusing mostly on rotational linear maps with some discussion of reflections. The first part deals with reflections. The second introduces the matrix associated with a rotational linear map: R_θ. Aside from a brief analysis of another linear map in $\mathbb{R}^2$, this rotation is what will be analyzed for the majority of the report. Overall, the motivation is to develop a geometric understanding of linear transformations and their matrix equivalents.

1 Linear Map Reflection

Our analysis of reflections of linear maps in $\mathbb{R}^2$ will be characterized by the following goal: finding a 2×2 matrix R such that the associated linear map $T_R : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors across the x-axis. The idea could easily be translated into a reflection across the y-axis and other lines in $\mathbb{R}^2$.

A reflection across the x-axis would maintain the x component of a vector $\vec{v} \in \mathbb{R}^2$ while reversing the sign of the y component. As such, we want to turn a vector

$$\vec{v} = \langle x, y \rangle \longrightarrow \langle x, -y \rangle$$

Linear maps from $\mathbb{R}^n \to \mathbb{R}^n$ can be described in terms of matrix multiplication.

$$T_R(\vec{v}) = R\vec{v}$$

We want to find $a, b, c, d \in \mathbb{R}$ such that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Multiplying the left side out produces:

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

We observe that $ax + by = x$ indicating that b must be zero for this statement to be true for all y. A similar argument may be applied to $cx + dy = -y$ such that the statement is true for all values of c, meaning c must be equal to 0 as well. These conclusions produce the following:

$$\begin{bmatrix} ax \\ dy \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

From here, $ax = x$ produces $a = 1$ and $dy = -y$ produces $d = -1$.

Combining the analysis for a, b, c, and d produces the following matrix that reflects vectors in $\mathbb{R}^2$ across the x-axis:

$$\boxed{R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

Therefore, $T_R(\langle x, y \rangle) = \langle x, -y \rangle$. Thus, T_R is a reflection across the x-axis.

2 Introduction to Rotational Linear Maps

We will begin our work with the following linear map. In order to work with the following matrix, fix an angle θ. We will sketch and describe the linear map T_θ associated to the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

i. Reducing to basis vectors

In order to study the effect of a linear maps in $\mathbb{R}^2$, we can reduce to studying the basis vectors. In $\mathbb{R}^n$, we have the standard basis

$$\mathcal{B} = \{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$$

Definition 2.1: An ordered set of vectors $\mathcal{B}$ is the standard basis of V if

1. $\mathcal{B} \subset$ V
2. span($\mathcal{B}$) = V
3. $\mathcal{B}$ is linearly independent

As such, if vector $\vec{v} \in \mathbb{R}^n$, it can be expressed as follows: $\vec{v} = \sum_{i=1}^{n} v_i \vec{e_i}$. This follows from part three the definition of a basis vector and the most integral to the character of the basis.

Specifically in $\mathbb{R}^2$, the standard basis is the following:

$$\mathcal{B} = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$$

Definition 2.2: A linear map $T : V \to W$is defined as follows for all $k \in \mathbb{N}, \alpha_i$

$$T\left(\sum_{i}^{k} \alpha_i x_i\right) = \sum_{i}^{k} \alpha_i T(x_i)$$

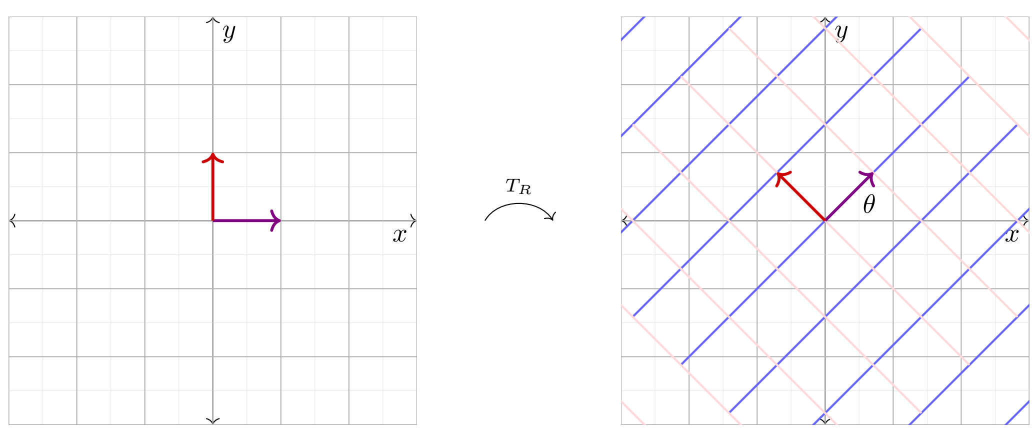
Following from the above definition of a linear map and the properties of the standard basis, we can see that the map can solely be applied to the standard basis vectors and adjusted through linear combinations. For the remainder of this problem and the following, we will analyze the effects of linear transforms on $\mathcal{B} = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$.

$$\vec{v} = x\vec{e_1} + ye_2$$
$$T(\vec{v}) = T(x\vec{e_1} + y\vec{e_2})$$
$$= xT(\vec{e_1}) + yT(\vec{e_2})$$

The above three lines were taken from the challenge report and make use of the previous definitions

ii. Sketch of Linear Map

The following is the sketch of the result of the R_θ on the basis vectors in $\mathbb{R}^2$, specifically where $\theta = \frac{\pi}{4}$:



The relevant LaTeXcode to create the above graphs was taken and modified from the challenge problem report

In this example, the linear map $R_{\theta=\frac{\pi}{4}}$ maintained the magnitude of the vectors while rotating it counterclockwise exactly $\frac{\pi}{4}$ radians.

iii. Proof of Rotational Nature of R_θ

First, analyzing $\vec{e}_1 = \langle 1, 0 \rangle$. Suppose the following: $\vec{u} = R_\theta \vec{e}_1$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

The magnitude of $\vec{u}$ using Pythagorean Theorem is $\cos^2\theta + \sin^2\theta$ which is equal to 1. This means that $\boxed{\text{the magnitude is preserved}}$. Furthermore, by definition, $\boxed{\text{the angle of rotation is } \sin^{-1}(\sin(\theta)) \text{ which is simply } \theta.}$

Analyzing $\vec{e}_2 = \langle 0, 1 \rangle$:

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

By the same reasoning as $\vec{e}_1$, the magnitude of $\vec{e}_2$ is preserved. In order to determine the rotation of $R_\theta \vec{e}_2$, we can simply check if it remains orthogonal to $R_\theta \vec{e}_1$. To do this, we will calculate the slope for both vectors.

$$m_1 = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

$$m_2 = -\frac{\cos(\theta)}{\sin(\theta)} = -\cot(\theta)$$

m_1 and m_2 are opposite reciprocals. By the definition of orthogonality in $\mathbb{R}^2$, $R_\theta \vec{e}_2$ remains perpendicular to $R_\theta \vec{e}_1$. Thus, the linear transformation associated with R_θ is a rotation by θ counterclockwise and preserves the magnitude of the original vector.

Theorem 2.3: The dot product between two vectors is zero when they are orthogonal to each other.

We could also make use of Theorem 2.3, taking the dot product of $R_\theta \vec{e}_1$ and $R_\theta \vec{e}_2$ to check if the two vectors are orthogonal. This is one of the many computational uses of the dot product.

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) = 0$$

Because the dot product of the two vectors is 0, they remain orthogonal, indicating that both were rotated by the same θ. *This idea was taken from office hours with Professor Wong.*

3 Linear Map T_S

Next, we will analyze the following matrix. Similar to part 2, we will sketch and describe the linear map T_S associated to the matrix.

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Following this description, we will find the eigenvectors of S.

i. Sketching and Describing T_S

We will continue to analyze the basis vectors $\vec{e}_1, \vec{e}_2$.

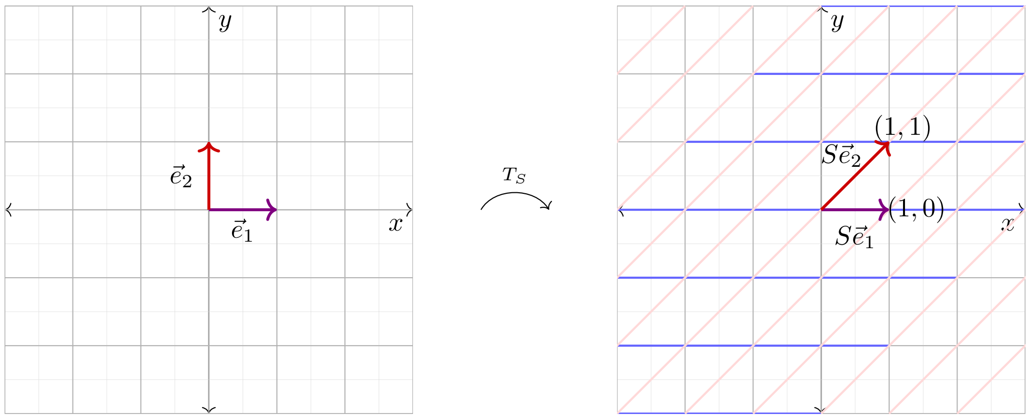
Performing the multiplication results in:

$$S\vec{e}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and...

$$S\vec{e}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This produces the following:



The relevant LaTeXcode to create the above graphs was taken and modified from the challenge problem report

Definition 3.1: The identity matrix is matrix associated with the linear transformation that leaves a vector unchanged. In $\mathbb{R}^2$ the identity matrix is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is worth noting the similarity of S and I. It helps to understand the effect of the transform. The extra 1 in the upper right corner of S essentially pulls $\vec{e}_2$ to the right while leaving $\vec{e}_1$ unchanged.

To describe the change in the magnitude of a vector and direction, we can analyze the effect on $\vec{v} = \langle v_1, v_2, \rangle$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_2 \end{bmatrix}$$

Thus, based on the resulting matrix, the magnitude is $\sqrt{(v_1 + v_2)^2 + v^2}$. The direction θ would be the $\sin^{-1}(\frac{\sqrt{(v_1+v_2)^2+(v_2)^2}}{v_2})$.

ii. Finding eigenvectors of S

Definition 3.2: Suppose H is the linear map associated to a matrix D. A non-zero vector $\vec{v}$ is an eigenvector of D and has an eigenvalue of λ if $D\vec{v} = \lambda\vec{v}$ where $\lambda \in \mathbb{R}$.

As described in the Challenge Report, solving for the roots of the characteristic polynomial $p(\lambda)$ in $\mathbb{R}^2$ produce eigenvalues.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$p(\lambda) = (a - \lambda)(d - \lambda) - bc$$

Plugging in the values of S into $p(\lambda)$ produces the following

$$p(\lambda) = (1 - \lambda)(1 - \lambda) - 0$$

$$\lambda = 1$$

1 is the eigenvalue of S. Now, in order to find the eigenvectors of S, we need to solve a system using our newly found eigenvalue.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$1)\ \ x + y = x$$

$$2)\ \ 0x + y = y$$

The above system has infinite solutions for x while y must equal to 0. This fits with the nature of eigenvectors as they can all be scaled. As such, the eigenvectors are:

$$\boxed{\alpha(\langle 1, 0\rangle) \text{ for all } \alpha \in \mathbb{R}}$$

In other words, the eigenvectors are all $\vec{v} \in \mathbb{R}^2$ parallel to $\langle 1, 0\rangle$

4 Real Eigenvalues of R_θ

We will return to R_θ that we analyzed in section 2. We are now focusing on finding any real eigenvalues of R_θ. More specifically, we will find the θ when such values exist. We will show clear steps to prove our results

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

From the characteristic polynomial, we have:

$$p(\lambda) = (a - \lambda)(d - \lambda) - bc$$

$$\begin{aligned} p(\lambda) &= (\cos(\theta) - \lambda)(\cos(\theta) - \lambda) + \sin^2(\theta) \\ &= \cos^2(\theta) - 2\lambda\cos(\theta) + \lambda^2 + \sin^2(\theta) \\ &= \lambda^2 + \lambda(-2\cos(\theta)) + 1 \end{aligned}$$

By the discriminant,

$$4\cos^2(\theta) - 4(1)(1) \geq 0$$

$$\cos^2(\theta) \geq 1$$

$$\cos(\theta) \geq 1 \text{ or } \cos(\theta) \leq -1$$

By the domain of the cosine function, $\cos(\theta) = 1, -1$. Therefore:

$$\boxed{\theta = \pi n \text{ for all } n \in \mathbb{N}}$$

Thus, only when $\theta = \pi n$ are there real eigenvalues for S. □

5 Eigenvectors of R_θ

The obvious next step after looking at eigenvalues is to look at the corresponding eigenvectors. Similar to question 2, we will fix an angle θ for analysis. It turns out there are only two cases to analyze, so we will literally fix θ to a specific value and then generalize it, analyzing the one other case in the process. Again, we are looking at the linear map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

i. Fix angle θ

Let's start by looking at the less obvious case that satisfies $\theta = \pi n \longrightarrow \theta = \pi$. Thus

$$R_{\theta=\pi} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Using our knowledge from definition 3.2, we can find our eigenvalue, which is needed to find a potential eigenvector.

$$\begin{aligned} p(\lambda) = 0 &= (-1-\lambda)(-1-\lambda) - 0 \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda+1)^2 \end{aligned}$$

$$\text{Thus, } \lambda = -1$$

After solving for λ, we need to solve the same system as in question 3.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

Plugging in our current values produces

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

This leaves us with

$$-x = x, -y = -y$$

Thus, as these equations are always true, the eigenvectors of $R_{\theta=\pi}$ are all $\langle a, b\rangle$ where $a, b \in \mathbb{R}$.

ii. Generalize to all θ

As proven in question 4, R_θ only has real eigenvalues at multiples of π. As such, only these values will have real eigenvectors.

Due to the nature of the sine and cosine function, the only other potential real eigenvector would occur at $\theta = 0$. From part 4, we know the eigenvalue we are considering here is 1. This would produce

$$R_{\theta=0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because this is the identity matrix, which by definition transforms a linear map to itself, the eigenvectors are the same as when $\theta = \pi$: all $\langle a, b\rangle$ where $a, b \in \mathbb{R}$.

Note: The same analysis using the identity matrix may have been used above. Simply factoring out the -1 would have produced the same result as in the case $R_{\theta=\pi}$.

In conclusion, where there exist real eigenvalues, the eigenvectors are the $\boxed{\text{complete vector space } \mathbb{R}^2.}$ Or more accurately, $\boxed{\mathbb{R}^2\backslash\{\vec{0}\}.}$ We cannot include the zero vector by the definition of a eigenvector, which includes a non zero clause. It is also easy to see some of the reasons why $\vec{0}$ would cause a problem. It would be an eigenvector for all linear maps and have undefined eigenvalues. *Saf pointed out the need to exclude the zero vector.* Imaginary eigenvectors will be expanded upon in question 6 with a more generalized version of R_θ.

Thus, there are always eigenvectors in this analysis, but only real eigenvectors when θ is an integer multiple of π.

6 Eigenvectors of M

Now we are going to look at a more general version of our trigonometric function based linear transformation. We will use the below matrix M. Let $a, b \in \mathbb{R}$ such that a and b are not both zero.

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We will now sketch and describe the linear map T_M associated to the matrix. Then, we will determine the eigenvectors of M.

i. Sketch and Describe T_M

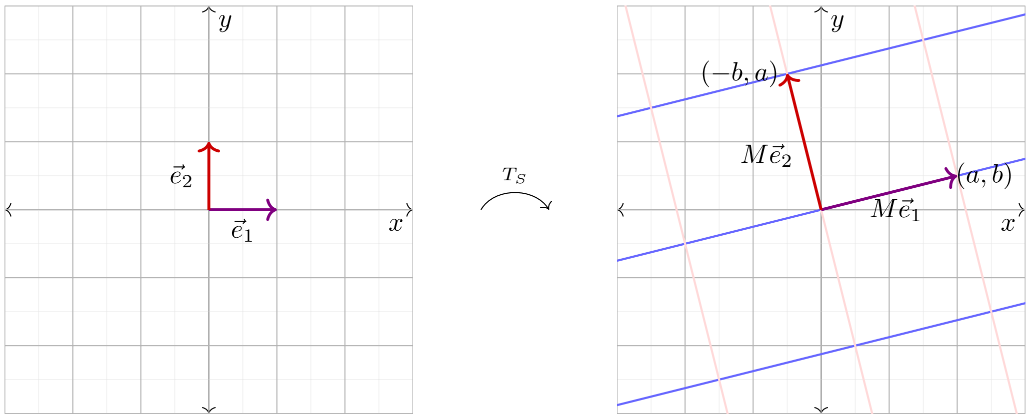
To map T_M, we must multiply M by the basis vectors in $\mathbb{R}^2$.

$$M\vec{e}_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

and...

$$M\vec{e}_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

This produces the following:



The relevant LaTeX code to create the above graphs was taken and modified from the challenge problem report

Specifically, the above sketch uses $a = 2$ and $b = 0.5$. Observe how, while the magnitude is not preserved, both basis vectors are rotated from their starting points. The magnitude of a vector $\vec{p}$ after T_M is applied to it is the following

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} ap_1 - bp_2 \\ bp_1 + ap_2 \end{bmatrix}$$

$$\|T_M(\vec{p})\| = \sqrt{(ap_1 - bp_2)^2 + (bp_1 + ap_2)^2}$$

For $\vec{e}_1$, it is clear that $M\vec{e}_1$ is a counterclockwise rotation of $\vec{e}_1$ by $\sin^{-1}\left(\frac{\sqrt{a^2+b^2}}{b}\right)$ from the values in the multiplication above. Using the slope analysis as in part 2–given the value of the dot product of the two is 0–it is clear that $\vec{e}_1$ remains orthogonal to $\vec{e}_2$, and thus is rotated by the same amount.

ii. Find eigenvalues of M

Using the characteristic polynomial produces

$$\begin{aligned} p(\lambda) = 0 &= (a - \lambda)(a - \lambda) + b^2 \\ &= a^2 - 2a\lambda + \lambda^2 + b^2 \\ &= \lambda^2 - \lambda(2a) + (a^2 + b^2) \end{aligned}$$

By the Quadratic Formula

$$\begin{aligned} \lambda &= \frac{2a \pm \sqrt{4a^2 - 4(1)(a^2 + b^2)}}{2} \\ &= \frac{2a \pm \sqrt{-4b^2}}{2} \\ &= \frac{2a \pm 2ib}{2} \\ &= a \pm bi \end{aligned}$$

From above, it is clear that our eigenvalues–and therefore our eigenvectors as well–will be imaginary unless $b = 0$.

iii. Find real eigenvectors of M

When an eigenvalue is imaginary, the eigenvector must also be imaginary. Thus, when looking for real eigenvectors, we must consider only real eigenvalues. Thus,

$$b = 0$$

This leaves a as our eigenvalue. Then, we will the equation below

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} x \\ y \end{bmatrix}$$

By definition, this equation is always true. As shown, a can be divided out and this reduces to definition 3.1, the definition of the identity matrix.

Thus, when $b = 0$, the eigenvectors are–same as part 5–the $\boxed{\text{complete vector space } \mathbb{R}^2 \text{ excluding } \vec{0}}$

iv. Find all eigenvector of M

Most of our work has been of vectors in $\mathbb{R}^2$. While imaginary eigenvectors are inherently not in $\mathbb{R}^2$, the math works our nicely and gives the same eigenvectors for all a, b not in our above special case, where $a \neq b$ and $b = 0$. In the following analysis, we will look at the other cases, when $b \neq 0$

To determine the imaginary eigenvectors, we will continue by solving our equation the same system as in previous questions, starting with $\lambda = a + bi$.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(a+bi) \\ y(a+bi) \end{bmatrix}$$

$$\begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = \begin{bmatrix} x(a+bi) \\ y(a+bi) \end{bmatrix}$$

This produces two separate equations

$$\begin{aligned} ax - by &= ax + xbi \\ -by &= xbi \\ y &= -xi \end{aligned}$$

$$\begin{aligned} bx + ay &= ay + ybi \\ bx &= ybi \\ x &= yi \end{aligned}$$

Note how the division by b is only possible because b is nonzero. Now, substituting

$$\begin{aligned} y &= -(yi)i \\ y &= y \\ y &= 1 \\ x &= 1(i) \\ x &= i \\ \langle x, y \rangle &= \langle i, 1 \rangle \end{aligned}$$

Observe, how like our other systems of equations in previous sections, this equation has infinite solutions given the nature of eigenvectors. This is why when we found the result $y = y$, we set y to 1 and continued.

Now, we must do the same analysis with $\lambda = a - bi$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(a-bi) \\ y(a-bi) \end{bmatrix}$$

$$\begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = \begin{bmatrix} x(a-bi) \\ y(a-bi) \end{bmatrix}$$

Again, this produces two separate equations

$$
\begin{aligned}
ax - by &= ax - xbi \\
-by &= -xbi \\
y &= xi
\end{aligned}
$$

$$
\begin{aligned}
bx + ay &= ay - ybi \\
bx &= -ybi \\
x &= -yi
\end{aligned}
$$

Substituting

$$
\begin{aligned}
x &= -(xi)i \\
x &= 1 \\
y &= 1(i) \\
y &= i \\
\langle x, y \rangle &= \langle 1, i \rangle
\end{aligned}
$$

Thus, eigenvectors are parallel to either $\langle i, 1 \rangle$ or $\langle 1, i \rangle$ for any $a, b \in \mathbb{R}$. Not only are they parallel by factor of a real λ, but they may also be scaled by a complex λ.

Summary

This report mostly analyzed how linear maps cause rotations in $\mathbb{R}^2$. We proved that R_θ led to a counterclockwise rotation while maintaining the magnitude of the original vector. We generalized that a linear map with matrix M that caused a similar rotation but did not maintain the magnitude of the original vector. We sketched and described these rotations and found both eigenvalues and eigenvectors when possible. In this report, I did the math myself with help from office hours. I did a peer review with Jack Hambidge and Lucas Schardt. Furthermore, I used the LaTeXcode from the challenge report to make the three sketches with some modifications. I received some minor feedback from Saf, mostly around the nuances of the complex eigenvectors in question 6.