Math 170A Homework 3

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Basics

Question 1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E_0, \mathcal{E}_0)$ and $(E_1, \mathcal{E}_1)$ be finite, discrete measurable spaces. Let $X : \Omega \to E_0$ be a measurable function, so that $(E_0, \mathcal{E}_0, \mathbb{P}_X)$ is a probability space. Let $h : E_0 \to E_1$ be an arbitrary function (which is necessarily measurable since $\mathcal{E}_0 = \mathcal{P}(E_0)$), and let $Y = h \circ X : \Omega \to E_1$ be the composition (which is thus measurable).
Prove that the measure $\mathbb{P}_1$ on E_1 induced by $Y : \Omega \to E_1$, when viewed as an E_1-valued random variable on the probability space $(\Omega, \mathcal{F}, P)$, is the same as the measure $\widetilde{\mathbb{P}}_1$ on E_1 induced by $h : E_0 \to E_1$, when viewed as an E_1-valued random variable on the probability space $(E_0, \mathcal{E}_0, \mathbb{P}_X)$.

We want to show that $\mathbb{P}_1 = \widetilde{\mathbb{P}}_1$. By definition, for $x \in E_1$

$$\mathbb{P}_1(\{x\}) \;=\; \mathbb{P}\big(Y^{-1}(\{x\})\big)$$

which is the probability measure of the preimage in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By the definition of Y, we then have that

$$\mathbb{P}\big(Y^{-1}(\{x\})\big) \;=\; \mathbb{P}\big(X^{-1} \circ h^{-1}(\{x\})\big)$$

But $h^{-1}(\{x\})$ may contain more than one point in E_0, thus, we can say that

$$X^{-1}\big(h^{-1}(\{x\})\big) \;=\; \bigcup_{y \in h^{-1}(\{x\})} X^{-1}(\{y\}),$$

and these events are each single disjoint elements. Therefore,

$$\mathbb{P}\big(X^{-1}(h^{-1}(\{x\}))\big) \;=\; \sum_{y \in h^{-1}(\{x\})} \mathbb{P}\big(X^{-1}(\{y\})\big).$$

By the definition of $\mathbb{P}_X$,

$$\mathbb{P}_X(\{y\}) \;=\; \mathbb{P}\big(X^{-1}(\{y\})\big).$$

Thus,

$$\mathbb{P}\big(Y^{-1}(\{x\})\big) \;=\; \sum_{y \in h^{-1}(\{x\})} \mathbb{P}_X(\{y\}) \;=\; \mathbb{P}_X\big(h^{-1}(\{x\})\big).$$

Finally, by definition of $\widetilde{\mathbb{P}}_1$,

$$\widetilde{\mathbb{P}}_1(\{x\}) \;=\; \mathbb{P}_X\big(h^{-1}(\{x\})\big).$$

Thus,

$$\mathbb{P}_1(\{x\}) \;=\; \widetilde{\mathbb{P}}_1(\{x\}),$$

and since this holds for every $x \in E_1$,

$$\mathbb{P}_1 = \widetilde{\mathbb{P}}_1.$$

□

Question 2

Now, let $X_1, X_2 : \Omega \to \mathbb{R}$ be discrete random variables with sets of values $V_{X_1}, V_{X_2} \subset \mathbb{R}$. Prove that for any function $h : \mathbb{R}^2 \to \mathbb{R}$ we have

$$E\big(h(X_1, X_2)\big) = \sum_{(x_1,x_2)\in V_{X_1}\times V_{X_2}} h(x_1, x_2)\, f_{X_1,X_2}(x_1, x_2)$$

Hint: Use Question 1, taking $E_0 = V_{X_1} \times V_{X_2}$, $X = (X_1, X_2)$, and $h = h|_{V_{X_1}\times V_{X_2}} : V_{X_1} \times V_{X_2} \to V_Y$.

Starting with the LHS, we have

$$\mathbb{E}(h(X_1, X_2)) = \sum_{y\in V_Y} y\mathbb{P}_1\big(\{h(X_1, X_2) = y\}\big)$$

But now, we can use a key equality from the previous part. We know that

$$\mathbb{P}_1 = \widetilde{\mathbb{P}}_1 \quad \implies \quad \mathbb{P}(Y^{-1}\{x\}) = \mathbb{P}_X(h^{-1}\{x\})$$

Hence, we can say that

$$\sum_{y\in V_Y} y\mathbb{P}\big(\{h(X_1, X_2) = y\}\big) = \sum_{y\in V_Y} y \sum_{(x_1,x_2)\in h^{-1}(\{y\})} \mathbb{P}_X\big((X_1, X_2) = (x_1, x_2)\big)$$

but from here, we can combine summands since we will cover every possible value in $V_X \times V_Y$ and express y as $h(x_1, x_2)$

$$\mathbb{E}(h(X_1, X_2)) = \sum_{(x_1,x_2)\in V_X\times V_Y} h(x_1, x_2)\mathbb{P}_X\big((X_1, X_2) = (x_1, x_2)\big)$$

Hence,

$$E\big(h(X_1, X_2)\big) = \sum_{(x_1,x_2)\in V_{X_1}\times V_{X_2}} h(x_1, x_2)\, f_{X_1,X_2}(x_1, x_2)$$

□

Conditional Distributions

Question 3

Let X and Y be two discrete random variables on an abstract sample space Ω, with possible values $V_X = \{1, 2, 3\}$ and $V_Y = \{1, 2\}$, and with joint probability mass function

$$f_{X,Y}(x, y) = \frac{x+y}{21}.$$

Calculate the values of the conditional distribution functions

$$g(x \mid y) = P(X = x \mid Y = y) \quad \text{and} \quad h(y \mid x) = P(Y = y \mid X = x),$$

for $x = 1, 2, 3$ and $y = 1, 2$.

We can first find the marginal probabilities for each X, Y, summing over all possible valuables of the opposite variable.

We can consider the $\mathbb{P}(X = x)$ for each $x \in V_X$. For $X = 1$, we have

$$\begin{aligned}\mathbb{P}(X = 1) &= \sum_{y \in V_Y} f_{X,Y}(1, y) \\ &= \frac{1+1}{21} + \frac{1+2}{21} \\ &= \frac{2}{21} + \frac{3}{21} \\ &= \frac{5}{21}\end{aligned}$$

For $X = 2$, we have

$$\begin{aligned}\mathbb{P}(X = 2) &= \sum_{y \in V_Y} f_{X,Y}(2, y) \\ &= \frac{2+1}{21} + \frac{2+2}{21} \\ &= \frac{3}{21} + \frac{4}{21} \\ &= \frac{7}{21}\end{aligned}$$

For $X = 3$, we have

$$\begin{aligned}\mathbb{P}(X = 3) &= \sum_{y \in V_Y} f_{X,Y}(3, y) \\ &= \frac{3+1}{21} + \frac{3+2}{21} \\ &= \frac{4}{21} + \frac{5}{21} \\ &= \frac{9}{21}\end{aligned}$$

We can consider the $\mathbb{P}(Y = y)$ for each $y \in V_Y$. For $Y = 1$, we have

$$\begin{aligned}\mathbb{P}(Y = 1) &= \sum_{x \in V_X} f_{X,Y}(x, 1) \\ &= \frac{1+1}{21} + \frac{2+1}{21} + \frac{3+1}{21} \\ &= \frac{2}{21} + \frac{3}{21} + \frac{4}{21} \\ &= \frac{9}{21}\end{aligned}$$

For $Y = 2$, we have

$$\begin{aligned}\mathbb{P}(Y = 2) &= \sum_{x \in V_X} f_{X,Y}(x, 2) \\ &= \frac{1+2}{21} + \frac{2+2}{21} + \frac{3+2}{21} \\ &= \frac{3}{21} + \frac{4}{21} + \frac{5}{21} \\ &= \frac{12}{21}\end{aligned}$$

Now, we can calculate the values of $g(x \mid y)$ and $h(y \mid x)$ using the fact that $\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(\{X=x\} \cap \{Y=y\})}{\mathbb{P}(Y=y)}$ and that in this formula, the numerator is simply our given function and the denominator is compromised of values we just calculated

Calculating $g(x \mid y) = \mathbb{P}(X = x \mid Y = y)$:

For $Y = 1$:

$$\mathbb{P}(X=1 \mid Y=1) = \frac{f_{X,Y}(1,1)}{\mathbb{P}(Y=1)} = \frac{\frac{2}{21}}{\frac{9}{21}} = \frac{2}{9},$$

$$\mathbb{P}(X=2 \mid Y=1) = \frac{f_{X,Y}(2,1)}{\mathbb{P}(Y=1)} = \frac{\frac{3}{21}}{\frac{9}{21}} = \frac{3}{9} = \frac{1}{3},$$

$$\mathbb{P}(X=3 \mid Y=1) = \frac{f_{X,Y}(3,1)}{\mathbb{P}(Y=1)} = \frac{\frac{4}{21}}{\frac{9}{21}} = \frac{4}{9}.$$

For $Y = 2$:

$$\mathbb{P}(X=1 \mid Y=2) = \frac{f_{X,Y}(1,2)}{\mathbb{P}(Y=2)} = \frac{\frac{3}{21}}{\frac{12}{21}} = \frac{3}{12} = \frac{1}{4},$$

$$\mathbb{P}(X=2 \mid Y=2) = \frac{f_{X,Y}(2,2)}{\mathbb{P}(Y=2)} = \frac{\frac{4}{21}}{\frac{12}{21}} = \frac{4}{12} = \frac{1}{3},$$

$$\mathbb{P}(X=3 \mid Y=2) = \frac{f_{X,Y}(3,2)}{\mathbb{P}(Y=2)} = \frac{\frac{5}{21}}{\frac{12}{21}} = \frac{5}{12}.$$

Calculating $h(y \mid x) = \mathbb{P}(Y = y \mid X = x)$:

For $X = 1$:

$$\mathbb{P}(Y=1 \mid X=1) = \frac{f_{X,Y}(1,1)}{\mathbb{P}(X=1)} = \frac{\frac{2}{21}}{\frac{5}{21}} = \frac{2}{5},$$

$$\mathbb{P}(Y=2 \mid X=1) = \frac{f_{X,Y}(1,2)}{\mathbb{P}(X=1)} = \frac{\frac{3}{21}}{\frac{5}{21}} = \frac{3}{5}.$$

For $X = 2$:

$$\mathbb{P}(Y=1 \mid X=2) = \frac{f_{X,Y}(2,1)}{\mathbb{P}(X=2)} = \frac{\frac{3}{21}}{\frac{7}{21}} = \frac{3}{7},$$

$$\mathbb{P}(Y=2 \mid X=2) = \frac{f_{X,Y}(2,2)}{\mathbb{P}(X=2)} = \frac{\frac{4}{21}}{\frac{7}{21}} = \frac{4}{7}.$$

For $X = 3$:

$$\mathbb{P}(Y=1 \mid X=3) = \frac{f_{X,Y}(3,1)}{\mathbb{P}(X=3)} = \frac{\frac{4}{21}}{\frac{9}{21}} = \frac{4}{9},$$

$$\mathbb{P}(Y=2 \mid X=3) = \frac{f_{X,Y}(3,2)}{\mathbb{P}(X=3)} = \frac{\frac{5}{21}}{\frac{9}{21}} = \frac{5}{9}.$$

Thus, we have calculated the values of each of those two functions, defining them on all necessary values in our set.

Question 4

Calculate the variances $\sigma^2(X)$ and $\sigma^2(Y)$, the covariance

$$\sigma(X,Y) = E\big[(X - \mu_X)(Y - \mu_Y)\big],$$

and the correlation coefficient

$$\rho_{X,Y} = \frac{\sigma(X,Y)}{\sigma_X \sigma_Y}.$$

Let's first calculate both $\mathbb{E}(X)$ and $\mathbb{E}(Y)$. This gives

$$\mu_X = \mathbb{E}(X) = \sum_{x \in V_X} x f_X(x) = \frac{5}{21} + \frac{14}{21} + \frac{27}{21} = \frac{46}{21}$$

and

$$\mu_Y = \mathbb{E}(Y) = \sum_{y \in V_Y} y f_Y(y) = \frac{9}{21} + \frac{24}{21} = \frac{33}{21}$$

Now, let's to find the variance, we also need $\mathbb{E}(X^2), \mathbb{E}(Y^2)$

For X:

$$\mathbb{E}(X^2) = \sum_{x \in V_X} x^2 f_X(x)$$

For Y:

$$\mathbb{E}(Y^2) = \sum_{y \in V_Y} y^2 f_Y(y)$$

$$= \frac{5}{21} + \frac{28}{21} + \frac{81}{21}$$

$$= \frac{114}{21}.$$

Thus, the variance of X is

$$\begin{aligned}\sigma^2(X) &= \mathbb{E}(X^2) - \mu_X^2 \\ &= \frac{114}{21} - \left(\frac{46}{21}\right)^2 \\ &= \frac{2394}{441} - \frac{2116}{441} \\ &= \frac{278}{441}.\end{aligned}$$

$$= \frac{9}{21} + \frac{48}{21}$$

$$= \frac{57}{21}.$$

Thus, the variance of Y is

$$\begin{aligned}\sigma^2(Y) &= \mathbb{E}(Y^2) - \mu_Y^2 \\ &= \frac{57}{21} - \left(\frac{33}{21}\right)^2 \\ &= \frac{1197}{441} - \frac{1089}{441} \\ &= \frac{108}{441} \\ &= \frac{12}{49}.\end{aligned}$$

Now, to find the covariance, consider

$$\mathbb{E}\big[(X - \mu_X)(Y - \mu_Y)\big] = \mathbb{E}(XY) - \mu_X \mathbb{E}(Y) - \mu_y \mathbb{E}(X) + \mu_X \mu_Y = \mathbb{E}(XY) - \mu_X \mu_Y.$$

This gives

$$\mathbb{E}(XY) = \sum_{x \in V_X} \sum_{y \in V_Y} xy\, f_{X,Y}(x, y).$$

Calculating term-by-term:

$$\begin{aligned}x = 1,\ y = 1: &\quad 1 \cdot 1 \cdot \frac{1+1}{21} = \frac{2}{21}, \\ x = 1,\ y = 2: &\quad 1 \cdot 2 \cdot \frac{1+2}{21} = \frac{6}{21}, \\ x = 2,\ y = 1: &\quad 2 \cdot 1 \cdot \frac{2+1}{21} = \frac{6}{21}, \\ x = 2,\ y = 2: &\quad 2 \cdot 2 \cdot \frac{2+2}{21} = \frac{16}{21},\end{aligned}$$

$$\begin{aligned}x = 3,\ y = 1: &\quad 3 \cdot 1 \cdot \frac{3+1}{21} = \frac{12}{21}, \\ x = 3,\ y = 2: &\quad 3 \cdot 2 \cdot \frac{3+2}{21} = \frac{30}{21}.\end{aligned}$$

Thus, summing these terms we have:

$$\mathbb{E}(XY) = \frac{2 + 6 + 6 + 16 + 12 + 30}{21} = \frac{72}{21}.$$

Therefore, the covariance is

$$\sigma(X, Y) = \mathbb{E}(XY) - \mu_X \mu_Y = \frac{72}{21} - \frac{46 \cdot 33}{21^2} = \frac{72}{21} - \frac{1518}{441} = \frac{1512 - 1518}{441} = -\frac{6}{441} = -\frac{2}{147}.$$

Finally, the correlation coefficient is given by

$$\rho_{X,Y} = \frac{\sigma(X, Y)}{\sigma_X \sigma_Y} = \frac{-\frac{2}{147}}{\left(\sqrt{\frac{278}{441}}\right)\left(\sqrt{\frac{108}{441}}\right)} = \frac{-\frac{2}{147}}{\frac{2\sqrt{834}}{147}} = -\frac{1}{\sqrt{834}}.$$