Math 170A Homework 1

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Monday, January 20, 2025

1 σ-Algebras

Question 1

Let $P = (P_i)_{i=1}^n$ be a finite partition of a (not necessarily finite) set Ω, that is, $P_i \subseteq \Omega$ for each $i = 1, \ldots, n$, $P_i \cap P_j = \emptyset$ for each $i \neq j$, and $\bigcup_{i=1}^n P_i = \Omega$.

1. Show that there exists a smallest σ-algebra $\mathcal{F}_P$ containing each of the subsets P_i.
2. How many subsets $A \subseteq \Omega$ are contained in $\mathcal{F}_P$?

To find the smallest σ-algebra $\mathcal{F}_P$ containing each of the subsets P_i, we need to consider all finite intersections of P_is as we know that intersections are empty. For each i, we either include or not include P_i in that specific union. Essentially, we have two options for each P_i, either part of the union or not. Thus, we have a total of 2^n elements in our σ-algebra $\mathcal{F}_P$. Every σ algebra with this partition would need to have all of these elements. Thus, the smallest σ-algebra $\mathcal{F}_P$ containing each of the subsets P_i is

$$\boxed{\mathcal{F}_P = \left\{ \bigcup_{i \in I} P_i \;\middle|\; I \subseteq \{1, 2, \ldots, n\} \right\}.}$$

generated by the finite partition, with $\boxed{2^n}$ elements. It is worth noting this is a σ algebra as complements are taken care of by including all unions that do not contain a specific element–which are accounted for in our definition above. Furthermore, it includes both the empty set and the full set. We also have that $\mathcal{F}_P \subset P(\Omega)$.

Question 2

Let $(E, \mathcal{E})$ be a finite, discrete measurable space, that is, E is a finite set equipped with $\mathcal{E} = \mathcal{P}(E)$, the σ-algebra given by the power set of E.
Prove that $f : (\Omega, \mathcal{F}_P) \to (E, \mathcal{E})$ is measurable if and only if the following holds: for each $i = 1, \ldots, n$, the restriction $f|_{P_i}$ of f to P_i is a constant function, that is,

$$f(\omega) = e_i \quad \text{for all } \omega \in P_i \text{ and some fixed } e_i \in E.$$

$(\implies)$ Assume that $f : (\Omega, \mathcal{F}_P) \to (E, \mathcal{E})$ is measurable. Let's pick an arbitrary P_i. Suppose for contradiction that f is not constant on this partition. Thus, for some $\omega, \omega' \in P_i$, we have that $f(\omega) = e_i \neq e_i' = f(\omega')$.

But, by the definition of measurability,

$$f^{-1}(\{e\}) \;=\; \bigcup_{i \in I_e} P_i \quad \text{and} \quad f^{-1}(\{e'\}) \;=\; \bigcup_{j \in I_{e'}} P_j.$$

Since $\omega \in P_i$ and $f(\omega) = e$, we get $\omega \in f^{-1}(\{e\})$. This forces both

$$P_i \;\subset\; f^{-1}(\{e\}) \quad P_i \;\subset\; f^{-1}(\{e'\})$$

However, this is a contradiction since both $f^{-1}(\{e\})$ and $f^{-1}(\{e'\})$ were required to be disjoint, as a function cannot map the same element to two different images by the definition of a function. Thus, we have that f must be constant on each partition P_i.

$(\Longleftarrow)$ Now, assume that f is constant on each partition P_i, i.e., for all $i \in I$ there is some $e_i \in E$ such that

$$f(\omega) \;=\; e_i \quad \text{for all } \omega \in P_i.$$

Now, we aim to show that $f^{-1}(A) \in \mathcal{F}_p$ for all $A \in \mathcal{E}$. Since $\mathcal{E} \; = P(E)$, we simply need to confirm this holds for all $A \subset E$. Consider some subset $A \in E$. If $e_i \in A$, then either $f^{-1}(e_i) = \phi$–which is an element of any sigma algebra–or there exists at least one $\omega \in \Omega$ such that $f(\omega) = e_i$. Because Ω is covered by disjoint subsets–i.e., partition elements P_i–we have that $\omega \in P_i$ for some P_i. Thus, for each $A \in E$, the preimage of every point is some P_i. Thus, for each set $A \subset E$, we have that

$$f^{-1}(A) = \bigcup_{i \in I_A} P_i$$

This, by definition, is clearly an element of $\mathcal{F}_p$. Thus, we have shown that $f^{-1}(A) \in \mathcal{F}_p$ for all $A \in \mathcal{E}$. This proves the reverse direction of this equivalence and therefore $f : (\Omega, \mathcal{F}_P) \to (E, \mathcal{E})$ is measurable if and only if for each $i = 1, \ldots, n$, the restriction $f|_{P_i}$ of f to P_i is a constant function, that is,

$$f(\omega) = e_i \quad \text{for all } \omega \in P_i \text{ and some fixed } e_i \in E.$$

$\square$

Question 3

Let $(E, \mathcal{E})$ be a measurable space and $X : \Omega \to E$. The σ-algebra on Ω generated by X is defined by

$$\mathcal{F}_X = \{X^{-1}(A) \mid A \in \mathcal{E}\}.$$

Let $(E_i, \mathcal{E}_i)$ be finite, discrete measurable spaces with $X_i : \Omega \to E_i$ for $i = 0, 1$, and let $\mathcal{F}_{X_i}$ be the σ-algebra generated by X_i for each $i = 0, 1$.
Prove that X_1 is measurable with respect to $\mathcal{F}_{X_0}$ if and only if there exists a function

$$h : E_0 \to E_1$$

such that $X_1 = h \circ X_0$.
The preceding is interpreted heuristically as the statement that X_0 determines X_1.

We will again prove this statement by considering both implications.

$(\Longrightarrow)$ Assume that X_1 is measurable with respect to $\mathcal{F}_{X_0}$. By the definition of measurability for discrete measurable spaces, we have that for all $A \in \mathcal{F}_{X_1}$, we have that $X_0^{-1}(A) \in \mathcal{F}_{X_0}$. This means that we need to show that $X^{-1}(A)$ must be some union of the elements that make up $\mathcal{F}_{X_0}$, i.e., $X_0^{-1}(\{e_0\})$ for all $e_0 \in E_0$.

This is the same definition as in question 1, as it is an equivalent definition of the smallest sigma algebra. Then, we simply need to show that X_1 is constant on each distinct $X_0^{-1}(e_0)$. Recall

$$\mathcal{F}_{X_0} = \{X_0^{-1}(A) \mid A \subset E_0\} \qquad \mathcal{F}_{X_1} = \{X_1^{-1}(A) \mid A \subset E_1\}$$

Suppose for contradiction that X_1 takes two different values $e_1, e_1' \in E_1$ on a one $X_0^{-1}(\{e_0\})$. But, then we have the same argument as in the previous question. By measurability, $X_1^{-1}(\{e_1\})$ and $X_1^{-1}(\{e_1'\})$ must both be in $\mathcal{F}_{X_0}$. This means they must both be some union of $X_0^{-1}(\{e\})$. However, again following the exact same argument, they both contain $X_0^{-1}(\{e_0\})$, which means the preimages are not disjoint, which is a contradiction of the definition of a function, mapping an input to only one output. Thus, X_1 is constant on each distinct preimage of X_0. This means that we can define the map for each $e_0 \in E_0$ such that $h(e_0) = e_1$ where e_1 is the singular value that X_1 takes on all elements of $X_0^{-1}(\{e_0\})$.

This produces h with this property as we can consider an $\omega \in \Omega$. Define $\epsilon_0 = X_0(\omega)$. Then, we have that $\omega \in X_0^{-1}(\{\epsilon_0\})$. Thus, we have that $h(X_0(\omega)) = X_1(\epsilon_0)$. Hence, as this is true for all $\omega \in \Omega$, we have that $h \circ X_0 = X_1$. This completes the forward direction.

$(\Longleftarrow)$ Now assume that there exists a function

$$h : E_0 \to E_1$$

such that $X_1 = h \circ X_0$. This means that for all $\omega \in \Omega$, $h(X_0(\omega)) = X_1(\omega)$. We aim to show measurability, that X_1 is measurable with respect to $\mathcal{F}_{X_0}$. Let $A \subset E_1$. Then, we aim to show that $X_1^{-1}(A) \in \{X_0^{-1}(A) \mid A \subset E_0\}$. However, by assumption, this is equivalent to showing that $X_0^{-1}(h^{-1}(A)) \in \{X_0^{-1}(A) \mid A \subset E_0\}$. However, because h is a function from $E_0 \to E_1$, we know that preimage of h is just a subset of E_0 and–since this has the discrete sigma algebra–we can just say that $B = h^{-1}(A) \subset E_0$. Then, the statement of measurability is trivially true. Thus, we have shown both directions and are done. □

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2 Probability Measures

Question 4

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, and let $X : \Omega \to E$ be a measurable function.
Prove that the function

$$P_X : \mathcal{E} \to \mathbb{R}$$

given by

$$P_X(A) = P\left(X^{-1}(A)\right)$$

defines a probability measure on $(E, \mathcal{E})$.

This question simply requires us to check the three conditions needed for P_X to be a measure. Before checking that, it is worth noting that the domain and codomain suffice here. We do not need to care about positive infinity as this measure will inherent the image of P which is just the typical $[0, 1]$. In each of the following, suppose $A \in \mathcal{E}$.

i **Non-negative:** Suppose $A \in \mathcal{E}$

$$P_X(A) = P(X^{-1}(A)) \text{ but } X^{-1}(A) \in \mathcal{F} \text{ by measurability}$$

Then, we immediately have that $P(X^{-1}(A)) \geq 0$ because P is a measure.

ii **Countably Additive:** Suppose that $A_1, A_2, \ldots \in \mathcal{E}$ and $A_i \cap A_j = \phi$ for $i \neq j$. Consider

$$\begin{aligned}
P_X(\bigsqcup_{i=1}^{\infty} A_i) &= P\left(X^{-1}\left(\bigsqcup_{i=1}^{\infty} A_i\right)\right) \\
&= P\left(\bigsqcup_{i=1}^{\infty} X^{-1}(A_i)\right) \qquad \text{by basic set theory} \\
&= \sum_{i=1}^{\infty} P\left(X^{-1}(A_i)\right) \qquad \text{since } P \text{ is a measure} \\
&= \sum_{i=1}^{\infty} P_X(A_i) \qquad \text{by the definition of } P_X
\end{aligned}$$

Thus, we have countable additivity.

iii **Empty Set has zero measure** $P_X(\phi) = P(X^{-1}(\phi)) = P(\phi) = 0$

Thus, we have shown that P_X is a measure. Lastly, to prove that it is a probability measure, consider the following

$$P_X(\mathcal{E}) = P(X^{-1}(\mathcal{E})) = P(\Omega) = 1$$

Thus, P_X is a probability measure. □