

# Math 120A Notes

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## Curves

**Definition** (Derivative / Jacobian). Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and fix  $a \in \mathbb{R}^m$ . We say  $f$  is (Fréchet) differentiable at  $a$  if there exists a linear map

$$df_a : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

such that

$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - df_a(h)\|}{\|h\|} = 0.$$

The linear map  $df_a$  is the **derivative** of  $f$  at  $a$ .

**Remark** (Matrix shape and action). Identify linear maps  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  with  $n \times m$  matrices. Then  $df_a \in \mathbb{R}^{n \times m}$  and for a column vector  $h \in \mathbb{R}^{m \times 1}$ ,

$$\underbrace{(df_a)}_{n \times m} \underbrace{h}_{m \times 1} = \underbrace{(df_a)(h)}_{n \times 1}.$$

**Proposition** (Coordinate description (Jacobian)). Write  $f(x_1, \dots, x_m) = (y_1, \dots, y_n) = (f_1, \dots, f_n)$ . If all the partial derivatives  $\frac{\partial f_i}{\partial x_j}(a)$  exist and  $f$  is differentiable at  $a$ , then  $df_a$  is represented by the **Jacobian matrix**

$$J_f(a) = [(df_a)_{ij}]_{i,j} \quad \text{with} \quad (df_a)_{ij} = \frac{\partial f_i}{\partial x_j}(a),$$

so that for  $h = (h_1, \dots, h_m)^\top$ ,

$$df_a(h) = J_f(a)h \quad \text{and} \quad \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \approx \begin{pmatrix} f_1(a) \\ \vdots \\ f_n(a) \end{pmatrix} + \underbrace{J_f(a)}_{n \times m} \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_m - a_m \end{pmatrix} \quad \text{near } a.$$

**Remark.** The mnemonic from coordinates is

$$(df_a)_{ij} = \left. \frac{\partial y_i}{\partial x_j} \right|_{x=a}.$$

**Proposition** (Chain Rule). Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  be differentiable, and set  $h = g \circ f$ . Then for any  $a \in \mathbb{R}^m$ ,

$$dh_a = dg_{f(a)} \circ df_a.$$

In matrix form,

$$d(g \circ f)_a = dg_{f(a)} df_a,$$

where  $dg_{f(a)} \in \mathbb{R}^{p \times n}$  and  $df_a \in \mathbb{R}^{n \times m}$ .

**Remark** (Coordinate Form). If  $x = (x_1, \dots, x_m)$ ,  $y = f(x) = (y_1, \dots, y_n)$ , and  $z = g(y) = (z_1, \dots, z_p)$ , then

$$\frac{\partial z_k}{\partial x_i} = \sum_{j=1}^n \frac{\partial z_k}{\partial y_j} \frac{\partial y_j}{\partial x_i},$$

or equivalently, using Einstein summation,

$$\frac{\partial z_k}{\partial x_i} = \frac{\partial z_k}{\partial y_j} \frac{\partial y_j}{\partial x_i}.$$

**Definition** (Regular Curve). A curve  $\alpha$  is **regular** if  $|\alpha'(t)| \neq 0$  for all  $t$ .

**Remark.** An irregular curve may fail to look smooth.

**Remark.** For a regular curve  $\alpha$ , the linear subspace spanned by  $\alpha'(t)$  gives the **tangent direction** at  $t$ .

**Definition** (Unit Tangent Vector). For a regular curve  $\alpha$ , the **unit tangent vector** is defined by

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}.$$

This requires  $\alpha$  to be regular.

**Remark.**  $T$  is an intrinsic property of the curve—it depends only on  $\text{Im}(\alpha) \subset \mathbb{R}^3$  and the direction of travel.

**Definition** (Reparametrization). Let  $I, J \subset \mathbb{R}$  be intervals, and let  $h : I \rightarrow J$  be a bijection with inverse  $g = h^{-1}$ . Assume  $h, g \in C^k$  and that  $h'(t) \neq 0$  for all  $t \in I$ . Then  $h$  (and  $g$ ) are called **reparametrizations**.

If  $\alpha : J \rightarrow \mathbb{R}^3$  is a  $C^k$  curve, the curve

$$\beta = \alpha \circ h : I \rightarrow \mathbb{R}^3$$

is called a **reparametrization of  $\alpha$** .

*Note.*  $h' \neq 0$  ensures that  $dg/ds \cdot dh/dt = 1$ , i.e.  $dg \circ dh = \text{Id}$  and  $dh \circ dg = \text{Id}$ .

**Proposition.** Let  $I, J \subset \mathbb{R}$  be intervals, and let  $h : I \rightarrow J$  be a  $C^k$  map. If  $h$  is surjective and  $h'(s) \neq 0$  for all  $s \in I$ , then  $h$  is a **reparametrization**.

**Remark.** Since  $h'$  never vanishes,  $h$  is locally monotone and hence injective, so  $h$  is bijective with  $h^{-1} \in C^k$ .

**Lemma** (Canonical Reparametrization by Arc Length). Every regular curve admits a canonical parametrization by arc length.

Let  $\alpha : J \rightarrow \mathbb{R}^3$  be a regular curve, where  $J \subset \mathbb{R}$  is an interval. Fix  $t_0 \in J$ , and define the **arc length function**

$$s(t) = \int_{t_0}^t |\alpha'(\tau)| d\tau.$$

Then  $s : J \rightarrow \text{Im}(s) = I$  is  $C^1$  and strictly increasing since

$$\frac{ds}{dt} = |\alpha'(t)| > 0.$$

Hence  $s$  is a  $C^1$  bijection with inverse  $t = s^{-1}$ .

The curve

$$\beta = \alpha \circ s^{-1} : I \rightarrow \mathbb{R}^3$$

is called the **arc-length parametrization** (or canonical reparametrization) of  $\alpha$ .

**Definition** (Frenet Frame for a Unit-Speed Curve). Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a regular curve parametrized by arc length, so  $|\alpha'(s)| = 1$ . Define

$$T = \alpha'(s), \quad T' = \alpha''(s).$$

The **curvature** is  $\kappa = |T'|$ , and if  $\kappa > 0$ , define the **principal normal vector**

$$N = \frac{T'}{|T'|}.$$

The **binormal vector** is

$$B = T \times N.$$

Then  $\{T, N, B\}$  form an orthonormal and positively oriented frame in  $\mathbb{R}^3$ .

Any vector  $v \in \mathbb{R}^3$  can be written uniquely as

$$v = aT + bN + cB,$$

with coefficients found by projection:

$$a = v \cdot T, \quad b = v \cdot N, \quad c = v \cdot B.$$

The orientation is positive since the matrix

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} \in \mathbb{R}^{3 \times 3}$$

is orthogonal with  $\det = +1$ , corresponding to the right-hand rule.

**Theorem** (Frenet–Serret Formulas for a Unit-Speed Curve). For a unit-speed curve  $\alpha : I \rightarrow \mathbb{R}^3$  with curvature  $\kappa > 0$  and torsion  $\tau$ , the frame vectors  $T, N, B$  satisfy

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Each entry is a function of the arc length parameter  $s$ . The matrix on the right-hand side is skew-symmetric and encodes the instantaneous rate of change of the orthonormal frame  $(T, N, B)$ .

**Definition** (Frenet Planes). For a unit-speed regular curve with Frenet frame  $(T, N, B)$ :

$$\text{Osculating plane} := \text{span}\{T, N\}, \quad \text{Normal plane} := \text{span}\{N, B\}, \quad \text{Rectifying plane} := \text{span}\{T, B\}.$$

**Remark** (Geometric Meaning of Torsion). From the Frenet–Serret equations,

$$B' = -\tau N \quad \text{so} \quad \tau = -\langle B', N \rangle.$$

Thus  $|\tau|$  is the *rate at which the binormal changes*, equivalently the instantaneous angular speed with which the osculating (or normal) plane rotates along the curve. In particular,  $\tau \equiv 0 \iff B' \equiv 0$ , so the osculating plane is constant and the curve is planar.

**Theorem** (Fundamental Theorem of Space Curves). Let  $I \subset \mathbb{R}$  be an open interval with  $0 \in I$ , and fix:

- a point  $x_0 \in \mathbb{R}^3$ ,
- a positive orthonormal frame  $(D, E, F)$  at  $x_0$ ,
- functions  $\bar{\kappa}, \bar{\tau}$  on  $I$  satisfying

$$\bar{\kappa} \in C^1(I), \quad \bar{\tau} \in C^0(I), \quad \bar{\kappa}(s) > 0 \text{ for all } s \in I.$$

Then there exists a unique unit-speed curve

$$\alpha : I \rightarrow \mathbb{R}^3$$

such that

$$\alpha(0) = x_0, \quad (T, N, B)(0) = (D, E, F), \quad \kappa = \bar{\kappa}, \quad \tau = \bar{\tau}.$$

**Theorem** (Picard–Lindelöf). Let  $I \subset \mathbb{R}$  be an open interval containing 0, and let

$$A : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$$

be a continuous function whose partial derivatives with respect to the spatial variables exist and are continuous on bounded subsets. Given an initial point  $c \in \mathbb{R}^n$ , there exists a unique function

$$\alpha : I \rightarrow \mathbb{R}^n$$

such that

$$\alpha(0) = c, \quad \frac{d\alpha(t)}{dt} = A(\alpha(t), t)$$

for all  $t \in I$ .

Equivalently, the solution satisfies the integral equation

$$\alpha(t) = c + \int_0^t A(\alpha(\tau), \tau) d\tau.$$

**Example.** Consider a curve with constant curvature and torsion:

$$\kappa = 5, \quad \tau = -3.$$

By the Fundamental Theorem of Space Curves, there exists a unique unit-speed curve (up to rigid motion) having these constants. Such a curve is a **circular helix**, expressible in the form

$$\alpha(t) = (r \cos t, r \sin t, ht),$$

where  $r$  and  $h$  are constants satisfying

$$\kappa = \frac{r}{r^2 + h^2}, \quad \tau = \frac{h}{r^2 + h^2}.$$

**Proposition** (Modified Frenet–Serret Formulas for Non–Unit-Speed Curves). Let  $\beta : I \rightarrow \mathbb{R}^3$  be a regular curve with velocity  $v = |\dot{\beta}(t)| > 0$ . Then the Frenet frame  $(T, N, B)$  satisfies

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad \text{so that} \quad \begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = v \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

That is,

$$\dot{T} = v\kappa N, \quad \dot{N} = -v\kappa T - v\tau B, \quad \dot{B} = v\tau N.$$

**Remark.** Since  $\dot{\beta} = vT$ , differentiation gives

$$\ddot{\beta} = \dot{v}T + v^2\kappa N,$$

so the tangential component of acceleration is  $\dot{v}$  and the normal component is  $v^2\kappa$ . Moreover,

$$\dot{\beta} \times \ddot{\beta} = v^3\kappa B.$$

**Theorem** (Frenet–Serret Formulas for Non–Unit Speed Curves). Let  $\vec{\beta}(t)$  be a  $C^3$  curve in  $\mathbb{R}^3$  with velocity  $\dot{\vec{\beta}}(t)$  and speed  $v = \|\dot{\vec{\beta}}\|$ . Then

$$\dot{\vec{\beta}} \times \ddot{\vec{\beta}} = v^3\kappa \vec{B},$$

where  $\kappa$  is the curvature and  $\vec{B}$  is the binormal vector. Hence,

$$\kappa = \frac{\|\dot{\vec{\beta}} \times \ddot{\vec{\beta}}\|}{v^3}, \quad \vec{B} = \frac{\dot{\vec{\beta}} \times \ddot{\vec{\beta}}}{\|\dot{\vec{\beta}} \times \ddot{\vec{\beta}}\|}.$$

Define  $\vec{T} = \frac{\dot{\vec{\beta}}}{\|\dot{\vec{\beta}}\|}$ ,  $\vec{N}$  the principal normal, and  $\vec{B}$  as above. Then  $\{\vec{T}, \vec{N}, \vec{B}\}$  form a right-handed orthonormal frame, with

$$\vec{N} = \vec{B} \times \vec{T}.$$

Differentiating, we have

$$\ddot{\vec{\beta}} = \dot{v}\vec{T} + v\dot{\vec{T}} = \dot{v}\vec{T} + v^2\kappa\vec{N}.$$

Taking one more derivative gives

$$\ddot{\vec{\beta}} = \ddot{v}\vec{T} + \dot{v}v\kappa\vec{N} + v^2\dot{\kappa}\vec{N} + v^3\kappa\dot{\vec{N}}.$$

Since  $\dot{\vec{N}} = -v\kappa\vec{T} - v\tau\vec{B}$  (where  $\tau$  is the torsion), we have

$$\ddot{\vec{\beta}} \cdot \vec{B} = v^3\kappa(-v\tau).$$

Hence, using the triple scalar product,

$$\tau = -\frac{\langle \ddot{\vec{\beta}}, \ddot{\vec{\beta}}, \ddot{\vec{\beta}} \rangle}{v^6\kappa^2}.$$

**Definition** (Tangent, Normal, and Binormal Vectors). Let  $\vec{\alpha}(s)$  be a smooth curve parameterized by arc length  $s$ . The *unit tangent vector* is defined as

$$\vec{T} = \frac{d\vec{\alpha}}{ds} = \vec{\alpha}'(s).$$

We can always define a unit *normal vector*  $\vec{N}$  perpendicular to  $\vec{T}$ , pointing toward the center of curvature of the curve.

The *binormal vector*  $\vec{B}$  is given by the cross product

$$\vec{B} = \vec{T} \times \vec{N}.$$

Here  $\vec{B}$  is a unit vector orthogonal to both  $\vec{T}$  and  $\vec{N}$ , completing a right-handed (positively oriented) orthonormal frame  $\{\vec{T}, \vec{N}, \vec{B}\}$ .

**Lemma** (Frenet–Serret formulas for a general parameterization). Let  $\beta : I \rightarrow \mathbb{R}^3$  be a regular  $C^3$  curve with  $\beta'(t) \neq \vec{0}$  for all  $t \in I$ . Then the Frenet–Serret apparatus is given entirely in terms of  $\beta$  and its derivatives as follows:

$$\begin{aligned} T(t) &= \frac{\beta'(t)}{\|\beta'(t)\|}, \\ \kappa(t) &= \frac{\|\beta'(t) \times \beta''(t)\|}{\|\beta'(t)\|^3}, \\ B(t) &= \frac{\beta'(t) \times \beta''(t)}{\|\beta'(t) \times \beta''(t)\|}, \\ N(t) = B(t) \times T(t) &= \frac{(\beta'(t) \times \beta''(t)) \times \beta'(t)}{\|\beta'(t) \times \beta''(t)\| \|\beta'(t)\|}, \\ \tau(t) &= \frac{[\beta'(t), \beta''(t), \beta'''(t)]}{\|\beta'(t) \times \beta''(t)\|^2}, \end{aligned}$$

where  $[\beta', \beta'', \beta''']$  denotes the scalar triple product  $\det(\beta', \beta'', \beta''')$ .

*Remarks.*

- These formulas are valid for any smooth non-unit-speed parameterization.
- The orientation convention used here assumes  $B = T \times N$ . If you adopt  $B = N \times T$  instead, the torsion acquires a minus sign.
- When  $\|\beta'(t)\| \equiv 1$  (unit speed), they reduce to the usual  $\kappa = \|T'\|$ ,  $\tau = -\langle B', N \rangle$ .

## Planar Curves

**Lemma** (Planar Curvature Direction). Suppose  $\vec{\alpha}(s)$  is a plane curve parameterized by arc length, and  $\kappa(s) \geq 0$  is its curvature. If  $\vec{T}'(s) = \kappa(s)\vec{N}(s)$  and  $\vec{T}'(s) \neq 0$  at some point, then  $\vec{N}(s)$  is purely normal (has no tangential component). Thus,  $\vec{N}(s) = \pm\hat{n}(s)$ , where  $\hat{n}$  is the unit normal to the curve.

Geometrically:

- $\vec{N} = \hat{n}$  when the curve turns *left* (counterclockwise).
- $\vec{N} = -\hat{n}$  when the curve turns *right* (clockwise).

**Definition** (Signed or Planar Curvature). The *planar curvature* of a plane curve  $\vec{\alpha}(s)$  is defined as

$$\vec{T}' = \kappa_p \vec{N} = \kappa_p \hat{n},$$

where  $\kappa_p$  is the *signed curvature*, given by

$$\kappa_p = \begin{cases} \kappa, & \text{if the curve turns left (CCW),} \\ -\kappa, & \text{if the curve turns right (CW).} \end{cases}$$

Hence,  $\kappa_p$  may be positive, negative, or zero, indicating left-turning, right-turning, or straight motion, respectively.

**Lemma** (Turning Angle Function). Let  $\alpha : I \rightarrow \mathbb{R}^2$  be a regular curve parametrized by arc length  $s$ , so that  $\|\alpha'(s)\| = 1$ . Then there exists a smooth function  $\theta : I \rightarrow \mathbb{R}$  (called the *turning angle function*) such that

$$\vec{T}(s) = \alpha'(s) = (\cos \theta(s), \sin \theta(s)).$$

*Proof.* Locally, for each  $s_0 \in I$ , since  $\vec{T}(s)$  is a unit vector in  $\mathbb{R}^2$ , it can be written as

$$\vec{T}(s) = (\cos \theta(s), \sin \theta(s))$$

for some smooth function  $\theta$  defined on a small interval around  $s_0$ . The angle  $\theta(s)$  is determined up to addition by integer multiples of  $2\pi$ .

We now show  $\theta$  can be defined smoothly on the entire interval  $I = [a, b]$ . Fix a point  $s_0 \in I$  and choose a reference angle  $\theta_0$  so that

$$\vec{T}(s_0) = (\cos \theta_0, \sin \theta_0).$$

On any small neighborhood of  $s_0$ , define  $\theta(s)$  smoothly so that  $\vec{T}(s) = (\cos \theta(s), \sin \theta(s))$ .

Since  $I$  is compact and the curve is smooth, we can cover  $I$  by finitely many overlapping intervals on which  $\theta$  is locally well-defined and smooth. The overlaps agree up to integer multiples of  $2\pi$ , and the orientation of the curve fixes this ambiguity, allowing a unique smooth global extension of  $\theta(s)$  across all of  $I$ .

Thus, a smooth turning angle function  $\theta : I \rightarrow \mathbb{R}$  exists and is unique up to addition of integer multiples of  $2\pi$ .  $\square$

**Proposition** (Planar Curvature and Normal Vector). Let  $\alpha : I \rightarrow \mathbb{R}^2$  be a unit-speed regular curve with

$$\vec{T}(s) = (\cos \theta(s), \sin \theta(s)).$$

Then the unit normal vector  $\vec{n}(s)$  and the planar curvature  $\kappa_p(s)$  are given by

$$\vec{n}(s) = (-\sin \theta(s), \cos \theta(s)), \quad \kappa_p(s) = \theta'(s).$$

*Proof.* Differentiating  $\vec{T}(s)$  with respect to  $s$ , we get

$$\vec{T}'(s) = (-\sin \theta(s) \theta'(s), \cos \theta(s) \theta'(s)) = \theta'(s) (-\sin \theta(s), \cos \theta(s)).$$

Since  $\vec{n}(s) = (-\sin \theta(s), \cos \theta(s))$  is a unit vector orthogonal to  $\vec{T}(s)$ , we have

$$\vec{T}'(s) = \kappa_p(s) \vec{n}(s) \quad \text{where} \quad \kappa_p(s) = \theta'(s).$$

The sign of  $\kappa_p$  encodes the orientation:  $\kappa_p > 0$  when the curve turns counterclockwise and  $\kappa_p < 0$  when it turns clockwise.  $\square$

**Example.** Let the curvature function be given by

$$\kappa_p(s) = \frac{1}{1+s^2}.$$

Then

$$\frac{d\theta}{ds} = \frac{1}{1+s^2} \quad \Rightarrow \quad \theta(s) = \arctan(s) + C.$$

Hence

$$\vec{T}(s) = (\cos \theta(s), \sin \theta(s)) = (\cos(\arctan s), \sin(\arctan s)) = \frac{1}{\sqrt{1+s^2}}(1, s).$$

The curve itself can then be recovered by integration:

$$\alpha(s) = \int \vec{T}(s) ds = \int \frac{1}{\sqrt{1+s^2}}(1, s) ds.$$

**Definition** (Unit-Speed Curve). A *unit-speed curve* is a smooth map

$$\alpha : I \rightarrow \mathbb{R}^2, \quad \text{such that} \quad \|\alpha'(s)\| = 1 \text{ for all } s \in I.$$

The tangent vector is

$$\vec{T}(s) = (\cos \theta(s), \sin \theta(s)),$$

where  $\theta(s)$  is the *angle function* of  $\alpha$ .

**Proposition** (Relation Between  $\theta$  and Curvature). For a unit-speed curve, the planar curvature  $\kappa_p(s)$  satisfies

$$\theta'(s) = \kappa_p(s), \quad \text{so that} \quad \theta(s) = \theta(s_0) + \int_{s_0}^s \kappa_p(\sigma) d\sigma.$$

Hence, for any two points  $s, t \in I$ ,

$$\theta(s) - \theta(t) = \int_t^s \kappa_p(\sigma) d\sigma,$$

which represents the total change in tangent angle between  $\alpha(t)$  and  $\alpha(s)$ .

**Definition** (Periodic Curve). A curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  is said to be *periodic with period*  $L > 0$  if

$$\alpha(s+L) = \alpha(s), \quad \forall s \in \mathbb{R}.$$

The smallest positive  $L$  for which this holds is called the *period* of  $\alpha$ .

**Remark.** This definition parallels that of periodic functions. When restricted to  $[0, L]$ , such a curve  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  is called a *unit-speed periodic curve of period  $L$* . These correspond to *closed curves* in  $\mathbb{R}^2$ , though not necessarily simple (they may self-intersect).

**Proposition** (Length of a Unit-Speed Periodic Curve). If  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  is unit-speed and periodic, then its period  $L$  coincides with its total arc length:

$$L = \int_0^L \|\alpha'(s)\| ds = \int_0^L 1 ds = L.$$

**Definition** (Index of a Closed Curve). Let  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  be a *closed, unit-speed, regular curve*, i.e.

$$\alpha(0) = \alpha(L), \quad \|\alpha'(s)\| = 1.$$

Then the *index* (or *turning number*) of  $\alpha$  is defined as

$$\text{ind}(\alpha) := \frac{1}{2\pi} \int_0^L \kappa_p(s) ds = \frac{1}{2\pi} (\theta(L) - \theta(0)),$$

where  $\kappa_p$  is the planar curvature and  $\theta(s)$  is the tangent angle satisfying  $\vec{T}(s) = (\cos \theta(s), \sin \theta(s))$ .

**Remark.** Since  $\alpha$  is closed, the initial and final tangent directions coincide:  $\theta(L)$  and  $\theta(0)$  represent the same direction in  $S^1$ . Hence, their difference must be an integer multiple of  $2\pi$ , i.e.

$$\theta(L) - \theta(0) = 2\pi n,$$

where  $n \in \mathbb{Z}$  is called the *index* of  $\alpha$ . Thus,

$$\text{ind}(\alpha) = n.$$

**Example.** For a simple counterclockwise circle,  $\theta(L) - \theta(0) = 2\pi$ , hence  $\text{ind}(\alpha) = +1$ . If the circle is traversed clockwise, then  $\theta(L) - \theta(0) = -2\pi$ , so  $\text{ind}(\alpha) = -1$ .

**Remark.** The index is also known as the *winding number* of the tangent vector around the unit circle  $S^1$ .

Moreover, since  $|\kappa_p| = |\theta'(s)|$ , one can also express the total (unsigned) curvature as

$$\int_0^L |\kappa_p(s)| ds = \int_0^L \left| \frac{d\theta}{ds} \right| ds.$$

This quantity measures the total amount of turning of the curve, regardless of orientation.

**Definition** (Simple Closed Curve). A *simple closed curve* is a closed curve  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  with no self-intersections. That is, for all  $s, t \in [0, L]$  with  $s \neq t$ , we have

$$\alpha(s) \neq \alpha(t).$$

Equivalently,  $\alpha$  is injective on  $[0, L)$ .

**Theorem** (Hopf Index Theorem). Every simple closed curve in the plane has index  $\pm 1$ .

**Theorem** (Jordan Curve Theorem). Let  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  be a simple closed curve. Then the complement  $\mathbb{R}^2 \setminus \text{im}(\alpha)$  has exactly two connected components: an *interior* and an *exterior*, with  $\alpha$  forming their common boundary.

**Definition** (Line Integral). Let  $\alpha : I \rightarrow \mathbb{R}^2$  be a smooth path, and let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be continuous. Then the line integral of the differential form  $f dx + g dy$  along  $\alpha$  is defined by

$$\int_{\alpha} f dx + g dy = \int_I (f(\alpha(s)) x'(s) + g(\alpha(s)) y'(s)) ds.$$

**Theorem** (Green's Theorem). Let  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  be a positively oriented, piecewise smooth, simple closed curve, and let  $R$  denote the region enclosed by  $\alpha$ . If  $f, g \in C^1(R)$ , then

$$\oint_{\alpha} f dx + g dy = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy.$$

**Remark.** The orientation of  $\alpha$  is positive when  $R$  (the interior region) always lies on the *left-hand side* of  $\alpha$  as one traverses it.

## Surfaces

**Definition** (Surface in  $\mathbb{R}^3$ ). A **coordinate patch** (or **surface patch**) is a smooth map

$$X : U \rightarrow \mathbb{R}^3,$$

where  $U \subset \mathbb{R}^2$  is open. We often assume  $U$  is connected. For each  $p \in U$ , there exists a disk  $D_r(p) \subset U$ .

**Definition** (Simple Patch). A coordinate patch  $X : U \rightarrow \mathbb{R}^3$  is called **simple** if it is injective.

**Definition** (Regular Surface). Let  $X : U \rightarrow \mathbb{R}^3$  be a coordinate patch, where  $U$  has coordinates  $(u_1, u_2)$  (also denoted  $(u, v)$  or  $(s, t)$ ). Define

$$X_1 = \frac{\partial X}{\partial u_1}, \quad X_2 = \frac{\partial X}{\partial u_2}.$$

We say that  $X$  is a **regular surface** if for every  $p \in U$ ,

$$X_1(p) \times X_2(p) \neq 0.$$

**Definition** (Tangent Vector). Let  $X : U \rightarrow \mathbb{R}^3$  be a regular surface with  $X_1 = \frac{\partial X}{\partial u_1}$  and  $X_2 = \frac{\partial X}{\partial u_2}$ . A **tangent vector** to the surface at  $X(p)$  is any vector in the subspace

$$\text{span}\{X_1(p), X_2(p)\}.$$

Equivalently, every tangent vector arises as the velocity vector of some curve on the surface passing through  $X(p)$ .

**Definition** (Unit Normal). The **unit normal vector** to the surface is defined (up to sign) by

$$\vec{n} = \frac{X_1 \times X_2}{\|X_1 \times X_2\|}.$$

**Example** (Graphs as Surfaces). Let  $f : U \rightarrow \mathbb{R}$  be a  $C^k$  function on an open set  $U \subset \mathbb{R}^2$ . Then the **graph** of  $f$  is a surface parameterized by

$$X(u_1, u_2) = (u_1, u_2, f(u_1, u_2)).$$

This  $X$  is a simple coordinate patch since it is injective, and  $X \in C^k$  whenever  $f \in C^k$ . We compute:

$$X_1 = (1, 0, f_1), \quad X_2 = (0, 1, f_2),$$

where  $f_i = \frac{\partial f}{\partial u_i}$ . Hence

$$X_1 \times X_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{vmatrix} = (-f_1, -f_2, 1),$$

which is nonzero everywhere, so the graph is a regular surface.

**Example.** Consider the map

$$x(\theta, \varphi) = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, \sin \varphi),$$

with parameter ranges

$$0 < \theta < 2\pi, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}.$$

This parameterization covers the sphere  $S^2$  except for the **missing poles** and the **meridian** where  $\theta = 0$ .

For a fixed value of  $\varphi$ , the curve

$$(\cos \theta \cos \varphi, \sin \theta \cos \varphi)$$

traces out a **circle of latitude** at height  $z = \sin \varphi$ .

Hence, the image of this map is

$$S^2 \setminus (\text{a meridian}).$$

**Definition** (Regular Surface Representation). Let  $U \subset \mathbb{R}^2$  be open. A **(coordinate) representation** or **parameterization** of a surface in  $\mathbb{R}^3$  is a  $C^k$  map

$$x : U \rightarrow \mathbb{R}^3$$

such that  $x$  is an immersion (i.e. the partial derivatives  $x_{u^1}, x_{u^2}$  are linearly independent everywhere).

If  $f, g \in C^k$  are two such parameterizations with overlapping domains, we say that they are **compatible** if

$$f \circ g^{-1}, g \circ f^{-1}$$

are  $C^k$  diffeomorphisms between open subsets of  $\mathbb{R}^2$ . In this case, we can write

$$y = x \circ g, \quad x = y \circ f,$$

where  $f, g \in C^k$  and  $f \circ g = \text{Id}_U, g \circ f = \text{Id}_U$ .

**Remark** (Normal Vector Field). For a regular parameterization  $x : U \rightarrow \mathbb{R}^3$ , the (unit) normal vector is defined by

$$n = \frac{x_{u^1} \times x_{u^2}}{\|x_{u^1} \times x_{u^2}\|},$$

up to sign.

Given a reparametrization  $y = x \circ g$ , we must verify that the orientation of the normal transforms properly:

$$\frac{y_{v^1} \times y_{v^2}}{\|y_{v^1} \times y_{v^2}\|} = \frac{x_{u^1} \times x_{u^2}}{\|x_{u^1} \times x_{u^2}\|} \quad \text{or} \quad -\frac{x_{u^1} \times x_{u^2}}{\|x_{u^1} \times x_{u^2}\|},$$

depending on whether the change of variables preserves or reverses orientation.

**Remark** (Jacobian Relations). If  $y = x \circ g$  with  $g = (u^1(v^1, v^2), u^2(v^1, v^2))$ , then by the chain rule,

$$y_{v^i} = \frac{\partial y}{\partial v^i} = \frac{\partial x}{\partial u^1} \frac{\partial u^1}{\partial v^i} + \frac{\partial x}{\partial u^2} \frac{\partial u^2}{\partial v^i} = \sum_{\alpha=1}^2 \frac{\partial x}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial v^i}.$$

This expresses how the tangent vectors transform under reparametrization.

**Proposition** (Transformation of the Normal under Reparametrization). Let  $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  and  $y : V \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be two coordinate patches representing the same surface, related by a smooth change of coordinates

$$y(v^1, v^2) = x(u^1(v^1, v^2), u^2(v^1, v^2)).$$

Then

$$y_{v^i} = \sum_{j=1}^2 x_{u^j} \frac{\partial u^j}{\partial v^i},$$

and the cross product of the tangent vectors transforms as

$$y_{v^1} \times y_{v^2} = (x_{u^1} \times x_{u^2}) \left( \frac{\partial u^1}{\partial v^1} \frac{\partial u^2}{\partial v^2} - \frac{\partial u^1}{\partial v^2} \frac{\partial u^2}{\partial v^1} \right).$$

The scalar factor in parentheses is the determinant of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial u^1}{\partial v^1} & \frac{\partial u^1}{\partial v^2} \\ \frac{\partial u^2}{\partial v^1} & \frac{\partial u^2}{\partial v^2} \end{pmatrix},$$

so that

$$y_{v^1} \times y_{v^2} = (x_{u^1} \times x_{u^2}) \det(J).$$

**Remark.** The determinant  $\det(J)$  determines whether the reparametrization preserves or reverses orientation. In particular, if  $\det(J) > 0$ , the orientation is preserved, whereas if  $\det(J) < 0$ , it is reversed.

By the chain rule,

$$d(g \circ f) = (dg)(df), \quad d(\text{Id}_U) = I_2,$$

which is consistent with the transformation of tangent vectors under coordinate changes.

**Remark** (Checking a Reparametrization). Let  $f : U \subset \mathbb{R}^2 \rightarrow V \subset \mathbb{R}^2$ . To check that  $f$  defines a valid **reparametrization**, verify that:

1.  $f$  is bijective,
2.  $f \in C^k$ ,
3.  $\det(df) \neq 0$  everywhere on  $U$ .

Then, by the **Inverse Function Theorem**,  $f^{-1}$  exists and is also  $C^k$ .

**Corollary.** If  $x : U \rightarrow \mathbb{R}^3$  is regular and  $y = x \circ f$ , where  $f$  is a reparametrization, then  $y$  is also regular. Moreover,

$$\frac{x_{u^1} \times x_{u^2}}{\|x_{u^1} \times x_{u^2}\|} = \pm \frac{y_{v^1} \times y_{v^2}}{\|y_{v^1} \times y_{v^2}\|}.$$

Hence, the **normal vector field** of a surface is an intrinsic property of the surface, defined up to a choice of orientation.

**Definition** (Surface Properties). For a regular parameterization  $x : U \rightarrow \mathbb{R}^3$ :

- The **unit normal vector** is

$$n = \frac{x_{u^1} \times x_{u^2}}{\|x_{u^1} \times x_{u^2}\|}.$$

- The **tangent plane** at  $P = x(u^1, u^2)$  is the two-dimensional vector space

$$T_P = \text{span}\{x_{u^1}, x_{u^2}\}.$$

Any tangent vector  $\vec{X} \in T_P$  can be written as

$$\vec{X} = X^1 x_{u^1} + X^2 x_{u^2}, \quad X^1, X^2 \in \mathbb{R}.$$

**Remark** (Tangent Vectors to Curves on Surfaces). A vector  $\vec{Y}$  is **tangent to the surface** at a point  $P = x(u^1, u^2)$  if it is the tangent vector of some regular curve on the surface passing through  $P$ .

That is, if  $\alpha : I \rightarrow U$  is a regular curve and  $\beta = x \circ \alpha$  is its image on the surface, then

$$\frac{d}{dt}(x \circ \alpha)(t) = x_{u^1} \frac{du^1}{dt} + x_{u^2} \frac{du^2}{dt}.$$

Thus, the tangent vector  $\vec{Y}$  lies in  $T_P$  and has coordinates

$$\vec{Y} = Y^1 x_{u^1} + Y^2 x_{u^2}, \quad Y^i = \frac{du^i}{dt}.$$

**Definition** (Surface in  $\mathbb{R}^3$ ). A **surface** is a subset  $S \subset \mathbb{R}^3$  that becomes a topological space by inheriting the subspace topology from  $\mathbb{R}^3$ .

A subset  $U \subset S$  is said to be open in  $S$  if and only if there exists an open set  $W \subset \mathbb{R}^3$  such that

$$U = S \cap W.$$

**Definition** (Coordinate Charts and Atlases). The surface  $S$  is a **regular surface** if for every point  $p \in S$ , there exists an open set  $W \subset S$  containing  $p$  and an open set  $U \subset \mathbb{R}^2$ , together with a map

$$x : U \rightarrow W,$$

such that:

1.  $x$  is a bijection onto  $W$ ;
2. both  $x$  and its inverse  $x^{-1} : W \rightarrow U$  are continuous;
3.  $x$  is of class  $C^k$  (often  $C^\infty$ );
4. the differential  $dx_q$  has rank 2 for all  $q \in U$  (i.e.,  $x$  is regular).

The map  $x$  is called a **coordinate chart** (or **local parameterization**).

The collection of all such pairs  $(U, x)$  that cover  $S$ ,

$$S = \bigcup_i x_i(U_i),$$

is called an **atlas** for  $S$ .

**Definition** (Compatibility Condition). Let  $x : U \rightarrow x(U)$  and  $y : V \rightarrow y(V)$  be two coordinate charts of  $S$  with overlapping domains such that  $x(U) \cap y(V) \neq \emptyset$ . Then the transition map

$$y^{-1} \circ x : x^{-1}(x(U) \cap y(V)) \rightarrow y^{-1}(x(U) \cap y(V))$$

must be a  $C^k$  diffeomorphism.

This ensures that  $x(U) \cap y(V)$  is open in  $S$ , and that the coordinate changes between charts are smooth.

## Key Theorems and Formulas

**Definition** (Metric Transformation Formula). For a change of coordinates  $u \mapsto v$ , the metric coefficients transform by

$$g_{ij} = \sum_{\alpha, \beta} \bar{g}_{\alpha\beta} \frac{\partial v^\alpha}{\partial u^i} \frac{\partial v^\beta}{\partial u^j}.$$

**Lemma** (Lemma 3.4; Determinant and Inverse Metric Identities). Let  $x : U \rightarrow \mathbb{R}^3$  be a coordinate patch, and let  $g_{ij}$  be the coefficients of the first fundamental form. Then:

- (a)  $g = \det(g_{ij}) = |x_1 \times x_2|^2$ ;
- (b) The entries of the inverse matrix satisfy

$$g^{11} = \frac{g_{22}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \quad g^{22} = \frac{g_{11}}{g};$$

- (c) For all  $i, j$ ,

$$\sum_{k=1}^2 g_{ik} g^{kj} = \delta_i^j.$$

**Proposition** (Differentiating a Curve in Local Coordinates). Let  $x : \mathcal{U} \rightarrow \mathbb{R}^3$  be a coordinate patch, and let

$$\gamma(s) = x(\gamma^1(s), \gamma^2(s)).$$

Then

$$\gamma'(s) = \sum_i x_i(\gamma(s)) (\gamma^i)'(s),$$

and differentiating once more gives

$$\gamma''(s) = \sum_{i,j} x_{ij}(\gamma(s)) (\gamma^i)'(s) (\gamma^j)'(s) + \sum_i x_i(\gamma(s)) (\gamma^i)''(s).$$

**Definition** (Normal and Geodesic Curvature). For a unit-speed curve  $\gamma$  on a surface, the decomposition

$$\gamma''(s) = \kappa_n(s) n(s) + \kappa_g(s) S(s)$$

defines two invariants:

- The *normal curvature*  $\kappa_n$  is the normal component of  $\gamma''$ .
- The *geodesic curvature*  $\kappa_g$  is the component of  $\gamma''$  in the direction  $S = n \times T$ .

**Lemma** (Curvature Decomposition). For a unit-speed curve on a surface,

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

**Definition** (Second Fundamental Form and Christoffel Symbols). For a simple surface  $x : \mathcal{U} \rightarrow \mathbb{R}^3$ :

- The coefficients of the *second fundamental form* are

$$L_{ij} = \langle x_{ij}, n \rangle.$$

- The *Christoffel symbols*  $\Gamma_{ij}^k$  (for  $1 \leq i, j, k \leq 2$ ) are defined by

$$\Gamma_{ij}^k = \sum_{\ell=1}^2 \langle x_{ij}, x_\ell \rangle g^{\ell k}.$$

**Proposition** (Proposition 4.2). Let  $x : \mathcal{U} \rightarrow \mathbb{R}^3$  be a simple surface. Then:

(a) **(Gauss's formulas)**

$$x_{ij} = L_{ij}n + \sum_k \Gamma_{ij}^k x_k. \quad (4-8)$$

(b) For any unit speed curve  $\gamma(s) = x(\gamma^1(s), \gamma^2(s))$ ,

$$\kappa_n = \sum_{i,j} L_{ij}(\gamma) (\gamma^i)'(\gamma^j)'. \quad (4-9)$$

(c)

$$\kappa_g S = \sum_k (\gamma^k)'' + \sum_{i,j} \Gamma_{ij}^k (\gamma^i)' (\gamma^j)' x_k. \quad (4-10)$$

**Proposition** (Proposition 4.3). Let  $x : \mathcal{U} \rightarrow \mathbb{R}^3$  be a coordinate patch with metric coefficients  $g_{ij}$ . Then

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^2 g^{k\ell} \left( \frac{\partial g_{\ell j}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^\ell} + \frac{\partial g_{i\ell}}{\partial u^j} \right). \quad (4-11)$$

*Proof.* Using the cyclic permutation of indices, we have

$$\frac{\partial g_{ij}}{\partial u^k} = \frac{\partial}{\partial u^k} \langle x_i, x_j \rangle = \langle x_{ik}, x_j \rangle + \langle x_i, x_{jk} \rangle, \quad (4-12)$$

$$\frac{\partial g_{ik}}{\partial u^j} = \langle x_{ij}, x_k \rangle + \langle x_i, x_{kj} \rangle, \quad (4-13)$$

$$\frac{\partial g_{jk}}{\partial u^i} = \langle x_{ji}, x_k \rangle + \langle x_j, x_{ki} \rangle. \quad (4-14)$$

Since  $x \in C^3$ , we have  $x_{ij} = x_{ji}$ . Combining (4-12), (4-13), and (4-14), we obtain

$$\frac{1}{2} \left( \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{jk}}{\partial u^i} \right) = \langle x_{ij}, x_k \rangle. \quad (4-15)$$

□

**Lemma** (Second Fundamental Form for a Surface of Revolution). For a surface of revolution parametrized by

$$x(u, v) = (r(u) \cos v, r(u) \sin v, z(u)),$$

the coefficients  $L_{ij}$  of the second fundamental form are given by

$$(L_{ij}) = \frac{1}{\sqrt{\dot{r}^2 + \dot{z}^2}} \begin{pmatrix} \ddot{r} \dot{z} - \dot{r} \ddot{z} & 0 \\ 0 & r \dot{z} \end{pmatrix}.$$

**Lemma.** Let  $M \subset \mathbb{R}^3$  be a regular surface with local parametrization

$$x : U \subset \mathbb{R}^2 \rightarrow M, \quad (u^1, u^2) \mapsto x(u^1, u^2),$$

and let  $\{x_1, x_2\}$  denote the coordinate tangent vectors, where  $x_i = \partial x / \partial u^i$ . Let  $I$  and  $II$  be the first and second fundamental forms of  $M$  in this chart, viewed as bilinear forms on the tangent space.

Then the Weingarten map (shape operator)  $L : T_p M \rightarrow T_p M$  at each point  $p = x(u^1, u^2)$  is characterized by

$$II(X, Y) = \langle LX, Y \rangle \quad \text{for all } X, Y \in T_p M,$$

and in the basis  $\{x_1, x_2\}$  its matrix is given by

$$[L] = I^{-1} II,$$

where  $I$  and  $II$  are identified with their  $2 \times 2$  matrices

$$I = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \quad II = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}.$$

**Lemma** (Transformation Law for the Second Fundamental Form). Let  $\bar{L}_{\alpha\beta}$  be the coefficients of the second fundamental form in a coordinate system  $v$ . Let  $f : v \mapsto u$  be a coordinate transformation. Then the coefficients  $L_{ij}$  in the  $u$ -coordinates satisfy

$$L_{ij} = \pm \sum_{\alpha,\beta} \bar{L}_{\alpha\beta} \frac{\partial v^\alpha}{\partial u^i} \frac{\partial v^\beta}{\partial u^j}.$$

**Definition** (Geodesic). A **geodesic** on a surface  $M$  is a unit speed curve on  $M$  whose geodesic curvature is identically zero.

**Proposition** (Proposition 5.1). A unit speed curve  $\gamma(s)$  in  $M$  is a geodesic if and only if

$$[n, T, T'] \equiv 0.$$

**Proposition** (Proposition 5.2). Let  $\gamma(s)$  be a unit speed curve,  $x$  a coordinate patch, and write  $\gamma(s) = x(\gamma^1(s), \gamma^2(s))$ . Then  $\gamma$  is a geodesic if and only if

$$\gamma^{k''} + \sum_{i,j} \Gamma_{ij}^k \gamma^{i'} \gamma^{j'} = 0, \quad k = 1, 2.$$

**Proposition** (Proposition 5.3). A unit speed curve  $\gamma(s)$  on a surface  $M$  is a geodesic if and only if  $\gamma''(s)$  is everywhere normal to the surface (i.e. a multiple of the normal to  $M$ ).

.....

**Proposition** (Proposition 5.5). Let  $M$  be a surface of revolution generated by the unit speed curve  $(r(t), z(t))$ . Then:

- (a) every meridian is a geodesic;
- (b) a circle of latitude is a geodesic if and only if the tangent  $x_1$  to the meridians is parallel to the axis of revolution at all points on the circle of latitude.

**Definition** (Second Fundamental Form). The **second fundamental form**  $\Pi$  on a surface  $M$  is the bilinear form on  $T_p M$  (for each  $p \in M$ ) given by

$$\Pi(X, Y) = \sum_{i,j} L_{ij} X^i Y^j,$$

where

$$X = \sum_i X^i x_i, \quad Y = \sum_j Y^j x_j \in T_p M.$$

**Proposition** (Proposition 7.1). Let  $M$  be a surface. Then:

- (a)  $\text{II}$  is a symmetric bilinear form on  $T_p M$  for each  $p \in M$ ;
- (b) if  $\gamma$  is a unit speed curve with tangent  $T$ , then  $\kappa_n = \text{II}(T, T)$ ;
- (c) if  $\alpha$  and  $\beta$  are regular curves with  $\alpha(0) = \beta(0)$  and whose velocity vectors are dependent at  $t = 0$ , then  $\alpha$  and  $\beta$  have the same normal curvature at  $t = 0$ .

**Definition** (Directional Derivative on a Surface). Let  $f$  be a differentiable function defined on an  $\varepsilon$ -neighborhood of a point  $P \in M$ . Let  $X \in T_P M$ , and let  $\alpha(t)$  be a curve on  $M$  such that

$$\alpha(0) = P, \quad X = \left( \frac{d\alpha}{dt} \right) (0).$$

The **directional derivative** of  $f$  in the direction  $X$  is

$$Xf = \left( \frac{d(f \circ \alpha)}{dt} \right) (0).$$

**Definition** (Weingarten Map). Let  $M$  be a surface and  $P \in M$ . The **Weingarten map** is the function

$$L : T_P M \longrightarrow \mathbb{R}^3, \quad L(X) = -Xn,$$

where  $n$  is the unit normal along  $M$ . (Since  $n$  is determined only up to sign, the same is true for  $L$ .)

**Proposition** (Proposition 7.6). Let  $M$  be a surface. Then:

- (a)  $L$  is a linear transformation from  $T_P M$  to  $T_P M$ ;
- (b) If  $L(x_k) = \sum_i L^i_k x_i$ , then

$$L^i_k = \sum_j L_{ij,k} g^{jl}.$$

**Proposition** (Weingarten's Equations on a Surface). On a surface  $M$  we have

$$n_j = - \sum_k L_j^k x_k. \tag{7-6}$$

**Lemma** (Lemma 7.10). If  $P \in M$  and  $X, Y \in T_P M$ , then

$$\text{II}(X, Y) = \langle L(X), Y \rangle = \langle X, L(Y) \rangle.$$

**Definition** (Principal Curvatures and Directions). At a point  $P$  of a surface  $M$ , the **principal curvatures** are the eigenvalues  $\kappa_1, \kappa_2$  of the Weingarten map  $L$ . The corresponding unit eigenvectors are the **principal directions** at  $P$ .

**Definition** (Umbilic Point). A point  $P$  on a surface  $M$  is an **umbilic** if

$$\kappa_1 = \kappa_2.$$

**Definition** (Line of Curvature). A **line of curvature** on a surface  $M$  is a curve whose tangent vector at each point is a principal direction at that point.

**Theorem** (Euler's Formula for Normal Curvature). Let  $Y$  be a unit tangent vector to  $M$  at  $P$ . Then the normal curvature in the direction  $Y$  is

$$\Pi(Y, Y) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta,$$

where  $\theta$  is the angle between  $Y$  and the principal direction  $X_{(1)}$  corresponding to  $\kappa_1$ .

**Definition** (Riemannian Curvature Tensor). The **Riemannian curvature tensor** with index  $(i, l, j, k)$  is defined by

$$R^i{}_{ljk} = \frac{\partial \Gamma^i_{lj}}{\partial u^k} - \frac{\partial \Gamma^i_{lk}}{\partial u^j} + \sum_p (\Gamma^i_{kp} \Gamma^p_{lj} - \Gamma^i_{jp} \Gamma^p_{lk}).$$

**Theorem** (Gauss's *Theorema Egregium*). The Gaussian curvature  $K$  of a surface is intrinsic.