Math 115B Homework 2

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1 Dual Basis

For each of the following vector spaces V and each (ordered) basis B, find an explicit formula for each vector in the dual basis B^*.

(a) $V = k^3, B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

We will find all three dual bases vectors at once. Consider how v_i^* acts on the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in k^3$. By the definition of a basis, we have that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, we have that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Taking the inverse, we have that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Expanding, we thus have that the dual basis vectors $B^* = \{v_1^*, v_2^*, v_3^*\}$ are as follows

$$v_1^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 = a - \frac{1}{2}b, \quad v_2^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_2 = \frac{1}{2}b, \quad v_3^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_3 = -a + c$$

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(b) $V = k[x]_{\leq 2}, B = \{1, x, x^2\}$.

We will find all three dual bases vectors at once. This example is more straightforward. Consider how v_i^* acts on the $a + bx + cx^2 \in k[x]_{\leq 2}$. By the definition of a basis, we have that

$$a + bx + cx^2 = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

We obviously then have that $\alpha_1 = a, \alpha_2 = b, \alpha_3 = c$.
Thus have that the dual basis vectors $B^* = \{v_1^*, v_2^*, v_3^*\}$ are as follows

$$v_1^*(a + bx + cx^2) = \alpha_1 = a, \quad v_2^*(a + bx + cx^2) = \alpha_2 = b, \quad v_3^*(a + bx + cx^2) = \alpha_3 = c$$

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2 Adjoint of a Transformation

Define some $f \in (\mathbb{R}^2)^*$ by $f\begin{pmatrix} x \\ y \end{pmatrix} = 2x + y$ and a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ via the formula

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ x \end{pmatrix}.$$

(a) Compute $T^*(f)$.

By definition, we have that $T^*(f) = f \circ T \quad$ for any $f \in (\mathbb{R}^2)^*$. Thus, consider

$$\begin{aligned} T^*(f)\begin{pmatrix} x \\ y \end{pmatrix} &= f \circ T\begin{pmatrix} x \\ y \end{pmatrix} \\ &= f \circ \begin{pmatrix} 3x + 2y \\ x \end{pmatrix} \\ &= 6x + 4y + x \\ &= 7x + 4y \end{aligned}$$

. . . .

(b) Compute $[T^*]_{E^*}$, where E is the standard ordered basis for $\mathbb{R}^2$ and $E^* = \{\vec{e}_1^*, \vec{e}_2^*\}$ is the dual basis, explicitly by finding scalars a, b, c, d such that $T^*(\vec{e}_1^*) = a\vec{e}_1^* + c\vec{e}_2^*$ and $T^*(\vec{e}_2^*) = b\vec{e}_1^* + d\vec{e}_2^*$.

Because E is the standard basis, the calculation of the dual basis is essentially trivial as in 1b and thus

$$E^* = \{v_1^*\begin{pmatrix} x \\ y \end{pmatrix} = x v_2^*\begin{pmatrix} x \\ y \end{pmatrix} = y\}$$

If we then consider

$$\begin{aligned} T^*(e_1^*)\begin{pmatrix} x \\ y \end{pmatrix} &= e_1^* \circ T\begin{pmatrix} x \\ y \end{pmatrix} \\ &= e_1^* \circ \begin{pmatrix} 3x + 2y \\ x \end{pmatrix} \\ &= 3x + 2y \end{aligned}$$

And similarly,

$$T^*(e_2^*)\begin{pmatrix}x\\y\end{pmatrix} = e_2^* \circ T\begin{pmatrix}x\\y\end{pmatrix}$$
$$= e_2^* \circ \begin{pmatrix}3x+2y\\x\end{pmatrix}$$
$$= x$$

Thus, by the definition of $[T^*]_{E^*}$, we have that

$$[T^*]_{E^*} = \begin{pmatrix}3 & 1\\2 & 0\end{pmatrix}$$

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(c) Compute $[T]_E$ and $([T]_E)^t$ and compare your result with your answer to the last question (you don't need to write anything about this comparison).

From the definition, since E is the standard basis, we can read off that

$$[T]_E = \begin{pmatrix}3 & 2\\1 & 0\end{pmatrix}$$

and easily calculate that

$$[T]_E^t = \begin{pmatrix}3 & 1\\2 & 0\end{pmatrix}$$

We can see that $[T^*]_{E^*} = [T]_E^t$, verifying a case of theorem 2.25 from class.

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3 Annihilators and Subspaces

Let V denote a finite-dimensional k-vector space. For any subset $S \subseteq V$, define the annihilator S^0 of S as

$$S^0 := \{f \in V^* : f(x) = 0 \text{ for all } x \in S\}.$$

(a) Prove that S^0 is a subspace of V^*.

We can easily check that S^0 contains the zero map as $f(x) = 0$ trivially satisfies the annihilation condition. Now to confirm that S^0 is closed under scalar multiplication and vector addition, suppose that $f, g \in S^0$ and $\lambda \in k$. Then, consider the following for all $x \in S$

$$(f + \lambda g)(x) = f(x) + \lambda g(x)$$
$$= 0 + \lambda 0$$
$$= 0$$

Thus, $f + \lambda g \in S^0$, and it is thus a subspace. $\square$

(b) If W is a subspace of V and $x \notin W$, prove that there exists some $f \in W^0$ such that $f(x) \neq 0$.

We can construct an f that has this property. We can leverage the finite dimensionality of V. By consequences of the Replacement Theorem, we can pick a basis β for W and note that $\beta \cup \{x\}$ must be linearly independent since $x \notin \text{span}(\beta)$. Thus, $\beta \cup \{x\}$ must be a basis for some subspace that contains W in V. We can continue to add vectors $\{v_1, \ldots v_n\}$ to $\beta \cup \{x\}$ until we have a basis for all of V–again by the Replacement Theorem. Let's call the basis $\gamma = \beta \cup \{x\} \cup \{v_1, \ldots v_n\}$. We can define $f \in V^*$ based on how it acts on each element of β–and by definition it will automatically be linear.

$$f(v) = \begin{cases} 0 & \text{if } v \in \beta \\ 1 & \text{if } v = x \\ 0 & \text{if } v \in \{v_1, \ldots v_n\} \end{cases}$$

Clearly, f is in W^0 since $f(v) = 0$ for all $v \in W$. However, f maps x to 1 which does not equal to zero in any field. $\square$

(c) In class, we constructed an isomorphism $\psi \colon V \to V^{**}$. Prove that $(S^0)^0 = \text{span}(\psi(S))$, where $\psi(S) := \{\psi(s) : s \in S\}$.

First note that

$$\psi(S) := \{\psi(s) : s \in S\} = \{\psi(s) : s \in S\}$$

Let's first show that $\text{span}(\psi(S)) \subset (S^0)^0$. Assume $h \in \text{span}(\psi(S))$. We know that

$$S^0 := \{f \in V^* : f(x) = 0 \text{ for all } x \in S\}.$$

And clearly, we have that

$$(S^0)^0 := \{g \in V^{**} : g(f) = 0 \text{ for all } f \in S^0\}.$$

By the definition of span, we have that

$$h = \sum_{i=1}^{n} \alpha_i \psi(s_i)$$

for $s_i \in S$. Then, if we consider h acting on some function $f \in V^*$,

$$h(f) = \sum_{i=1}^{n} \alpha_i \psi(s_i)(f) = \sum_{i=1}^{n} \alpha_i f(s_i)$$

which we get by the definition of ψ from class. Then, $h(f) = 0$ whenever $f(s_i) = 0$ for all $s \in S$. Thus, $h \in (S^0)^0$ and therefore, $\text{span}(\psi(S)) \subset (S^0)^0$.

As for showing $(S^0)^0 \subset \text{span}(\psi(S))$, we can simply reverse the logic. Suppose $h \in (S^0)^0$. Then, $h(f) = 0\ \forall f \in S^0$ by definition. Because we know there is natural isomorphism ψ, there exists a unique vector $v \in V$ such that $\psi(v)(f) = h(f)$. Then, we have that $f(v) = 0\ \forall f \in S^0$. We need to then show that v is in the span of S. However, this is immediate from part (b). We know that by the definition of S^0, f must vanish on all of the vectors of S. Furthermore, it must vanish on all the vectors in the span S. By (b), we know that it is impossible for v to vanish and not be in the span. Thus, $v \in \text{span}(S)$. Lastly, by linearity, since $h = \psi(v)$, we have that

$$h = \psi\left(\sum_{i=1}^{n} a_i s_i\right) \implies \sum_{i=1}^{n} a_i \psi(s_i)$$

Therefore, $h \in \text{span}(\psi(S))$. We have shown both inclusions, and thus $(S^0)^0 = \text{span}(\psi(S))$

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(d) For subspaces W_1 and W_2 of V, prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

($\implies$) Assume that $W_1 = W_2$. This directly is trivially true. We have that

$$W_1^0 := \{f \in V^* : f(x) = 0 \text{ for all } x \in W_1\} \qquad W_2^0 := \{f \in V^* : f(x) = 0 \text{ for all } x \in W_2\}$$

Again, trivially, this is true by simply relabeling.
($\impliedby$) Assume that $W_1^0 = W_2^0$. Then, we can take the annihilator of both sides, producing

$$(W_1^0)^0 = (W_2^0)^0$$

By (c), this is equivalent to

$$\text{span}(\psi(W_1)) = \text{span}(\psi(W_1))$$

But, we can remove the ψ by linearity–pulling it outside the span–and then ignore it by injectivity. Then, we just have that

$$\text{span}(W_1) = \text{span}(W_2)$$

However, a subspace is invariant under span. Thus,

$$W_1 = W_2$$

Therefore, $W_1 = W_2$ if and only if $W_1^0 = W_2^0$. □

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(e) For subspaces W_1 and W_2, prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

Let's first show that $(W_1 + W_2)^0 \subset W_1^0 \cap W_2^0$. Suppose that $f \in (W_1 + W_2)^0$. Then, for $v \in W_1 + W_2$, we have that $f(v) = 0$. Suppose that $v_1 \in W_1$ and that $v_2 \in W_2$. Note, $v_1, v_2 \in W_1 + W_2$ since we can just tack the zero vector from the other subspace. Thus, f also vanishes on any element of W_1 and W_2. Hence, $f \in W_1$ and $f \in W_2$. Thus, $(W_1 + W_2)^0 \subset W_1^0 \cap W_2^0$.

Now, let's show that $W_1^0 \cap W_2^0 \subset (W_1 + W_2)^0$. Suppose that $f(v_1) = 0$ for all $v_1 \in W_1$, and similarly, $f(v_2) = 0$ for all $v_2 \in W_2$. Suppose that $v \in W_1 + W_2$. By the definition of $W_1 + W_2$, we know $v = v_1' + v_2'$ for $v_1' \in W_1$ and $v_2' \in W_2$. Then, we know that $f(v) = f(v_1') + f(v_2') = 0 + 0 = 0$. Thus, $f \in (W_1 + W_2)^0$. Hence, $W_1^0 \cap W_2^0 \subset (W_1 + W_2)^0$. Since we have shown both inclusions, we have that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

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4 Dimension of Annihilator

Prove that if W is a subspace of V, then

$$\dim(W) + \dim(W^0) = \dim(V).$$

(For one point less: you may assume that $\dim(V) < \infty$.)

I will attempt to prove this in the general case–including infinite dimensions–which relies on the idea that every infinite dimensional vector space has a basis. Suppose that $\{v_a \mid a \in A\}$ for an index set A–where A is not necessarily countable–is a basis for W. We can further extend this to a basis $\{v_a \mid a \in A\} \cup \{w_b \mid b \in B\}$. Then, we can construct a basis $\{v_a^* \mid a \in A\} \cup \{w_b^* \mid b \in B\}$ where we still map $v_a^*(v_a) = 1$ and zero otherwise. However, this may be a problem in the infinite case, as this is not necessarily the full basis of V^*. Either way, we can still claim that $\{w_b^* \mid b \in B\}$ set up as as described is a basis for W^0.

Now consider $\{w_b^* \mid b \in B\}$. I claim this is a basis for W^0 as we will show. By definition, these vectors are linearly independent as they are a subset of a linearly independent set (i.e., our constructed basis). (This logic still holds in the infinite case when we do not have a basis for $V^*$of the same cardinality as V as what we have above for a basis of V^* would just be a subset of the actual bases.) Then, we just need to show every element in W^0 is in the span of these vectors. Suppose $f \in W^0$. Then, we know we can write f generally as an element in $V*$. Thus, we have that

$$f = \sum_{a \in A} f(v_a) v_a^* + \sum_{b \in B} f(v_b) v_b^*$$

However, we have that $f(v_a) = 0 \; \forall a \in A$. Thus,

$$f = \sum_{b \in B} f(v_b) v_b^*$$

Hence, $f \in \operatorname{span}(\{w_b \mid b \in B\})$. Therefore, we have that $|W^0| + |W| = |V|$. Thus, we have shown for any case,

$$\dim(W) + \dim(W^0) = \dim(V).$$

However, another issue potentially arises with this writing of $f = \sum_{a \in A} f(v_a) v_a^* + \sum_{b \in B} f(v_b) v_b^*$ in the infinite case because not any f can necessarily be written this way. Yet, any f that can potentially be in W^0 would be in this form.

□

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5 Kernel and Range Annihilator

Suppose that W is a finite-dimensional vector space and $T \colon V \to W$ is a linear transformation. Prove that $\ker(T^*) = \operatorname{im}(T)^0$.

Let's first show that $\ker(T^*) \subset \operatorname{im}(T)^0$. Suppose $f \in \ker(T^*)$. This means that $f \in W^*$ and $T^*(f) = f \circ T = \vec{0}_{\text{map}}$. This means that $f(x) = 0$ where $x \in \operatorname{Im}(T)$. Thus, $f \in \operatorname{Im}(T)^0$ as it annihilates everything in the image. Thus, $\ker(T^*) \subset \operatorname{im}(T)^0$.

Now suppose that $\operatorname{Im}(T)^0 \subset \ker(T^*)$. Suppose that $f \in \operatorname{Im}(T)^0$. Then, by definition, $f(x) = 0 \; \forall x \in \operatorname{Im}(T)$. Written another way, $f \circ T(x) = 0$ for all $x \in V$. This is by definition saying, $T^*(f) = \vec{0}_{\text{map}}$. Hence, $f \in \ker(T^*)$.

We thus have both inclusions and therefore $\ker(T^*) = \operatorname{im}(T)^0$. □

6 Eigenvalues and Eigenspaces

Let R denote the 3×3 real matrix

$$R = \begin{pmatrix} -3 & -3 & -4 \\ 2 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find all eigenvalues of R. For each eigenvalue, compute the corresponding eigenspace.

We need to find the determinant of $R - \lambda I$:

$$\det(R - \lambda I) = \begin{vmatrix} -3-\lambda & -3 & -4 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{vmatrix}.$$

Expanding along the bottom row, we have

$$\det(R - \lambda I) = (-1-\lambda) \cdot \begin{vmatrix} -3-\lambda & -3 \\ 2 & 2-\lambda \end{vmatrix}.$$

This gives

$$\det(R - \lambda I) = (-1-\lambda)(\lambda^2 + \lambda) = 0.$$

Solving for λ:

$$(\lambda)(\lambda+1)^2 = 0 \quad \Rightarrow \quad \lambda = -1, \quad \lambda = 0.$$

$\lambda = -1$ has multiplicity 2.

Let's now find the eigenvalues.

Consider the $\lambda = 0$ eigenvalue Solving the equation $R\vec{x} = \vec{0}$, we can row reduce

$$\left[\begin{array}{ccc|c} -3 & -3 & -4 & 0 \\ 2 & 2 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus, we have that one eigenvector is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Thus, the eigenspace for $\lambda = 0$ is:

$$E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Consider the $\lambda = 0$ eigenvalue Solving the equation $(R + I)\vec{x} = \vec{0}$, we can row reduce

$$\left[\begin{array}{ccc|c} -2 & -3 & -4 & 0 \\ 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this, there are two L.I. eigenvectors

$$\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, the eigenspace for $\lambda = -1$ is:

$$E_{-1} = \text{span}\left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

All together,

→ The eigenvalues of R are:

$$\lambda = 0, \quad \lambda = -1 \quad \text{(with multiplicity 2)}.$$

→ The corresponding eigenspaces are:

$$E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and

$$E_{-1} = \text{span}\left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

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7 Diagonalizable Transformation

For the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by the formula

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix},$$

find a basis B of $\mathbb{R}^2$ such that $[T]_B$ is diagonal (and prove your answer is correct).

We can write this in matrix form, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvectors of this matrix and its corresponding eigenvalues are $\lambda_1 = 2$ corresponding to $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\lambda_2 = 3$ corresponding to $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. I claim that a basis

$$\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

makes $[T]_B$ diagonal. Then, to check, $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Clearly, we then have that

$$[T]_\beta = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

which verifies that $[T]_\beta$ is a diagonal matrix of eigenvalues. □

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8 Invariant Subspaces

Given some vector space V and a linear endomorphism $T\colon V \to V$ (i.e., a linear transformation with the same domain and codomain, often also called a linear operator), we define a T-invariant subspace of V to be a subspace $W \subseteq V$ such that $T(W) \subseteq W$. For each of the following linear endomorphisms, determine whether the given subspace is T-invariant subspace of V.

(a) $V = \mathbb{R}[x], T(f(x)) = f'(x), W = \mathbb{R}[x]_{\leq 2}$.

This is indeed a T invariant subspace as the derivative of $ax^2+bx+c \in \mathbb{R}[x]_{\leq 2}$ is $T(ax^2+bx+c) = 2ax+b$. This remains in $\mathbb{R}[x]_{\leq 2}$. □

. . . .

(b) $V = \mathbb{R}[x], T(f(x)) = xf(x), W = \mathbb{R}[x]_{\leq 2}$.

This is not a T invariant subspace. We can consider a counterexample. Take $f(x) = x^2 \in \mathbb{R}[x]_{\leq 2}$. Clearly, $x^3 \notin \mathbb{R}[x]_{\leq 2}$. □

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(c) $V = k^3, T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+x_2+x_3 \\ x_1+x_2+x_3 \\ x_1+x_2+x_3 \end{pmatrix}, W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = x_2 = x_3 \right\}$.

For any $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in k^3$, let $y = x_1 + x_2 + x_3$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+x_2+x_3 \\ x_1+x_2+x_3 \\ x_1+x_2+x_3 \end{pmatrix} = \begin{pmatrix} y \\ y \\ y \end{pmatrix}$$

Clearly this satisfies being in W. □

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(d) V is the set of all continuous functions $[0,1] \to \mathbb{R}$, $T(f(t)) = \left(\int_0^1 f(x)dx\right)t, W = \{f \in V : f(t) = at + b \text{ for some } a, b \in \mathbb{R}\}$.

This is indeed a T-invariant subspace. Consider $\int_0^1 at + bdt$. This produces $\frac{a}{2} + b$. Multiplying by t gives something of the form $a't + b'$ where $a' = \frac{a}{2} + b$ and $b' = 0$. Thus, this is invariant.

. . . .

(e) $V = k^{2\times 2}, T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A, W$ is the subspace of symmetric 2×2 matrices, i.e., those 2×2 matrices satisfying $A^t = A$.

This is not invariant. Consider $T\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

This matrix is clearly not symmetric. Thus, it is not invariant. □