Math 115AH Homework 3

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**When not explicitly mentioned, α_i, a_i are scalars in the field of whichever vector space we are working with*

1 Computation Problems

Textbook 1.6 3(b)

$$\{1 + 2x + x^2, 3 + x^2, x + x^2\}$$

where $a, b, c \in \mathbb{R}$

$$a(x^2 + 2x + 1) + b(x^2 + 3) + c(x^2 + x) = \vec{0}$$

Thus, we have

$$ax^2 + bx^2 + cx^2 = 0x^2 \qquad 2ax + cx = 0x \qquad a + 3b = 0$$

which gives us

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}\right) = -4$$

Thus, our set is linearly independent. It also has sufficient degrees of freedom so it spans $P_2(\mathbb{R})$

Textbook 1.6 6. Here are three basis for each F^2 and $M_{2\times 2}(F)$

For F^2, where $1 = 1_F, 0 = 0_F$, we can have

$$\{\langle 1, 0\rangle, \langle 0, 1\rangle\} \qquad \{\langle -1, 0\rangle, \langle 0, -1\rangle\} \qquad \{\langle 1, 1\rangle, \langle 0, 1\rangle\}$$

And for $M_{2\times 2}(F)$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

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A basis for the set of $n \times n$ upper triangular matrices is

$$\beta = \{E_{ij} \mid 1 \le i \le j \le n\}$$

where E_{ij} corresponds to the standard basis vectors for $M_{n\times n}(F)$. The dimension of the vector space is the number of vectors in the basis. If we think about the matrix by each of its rows, we will have n elements for the first row, $n - 1$ for the second, and so on until we only have one element in the final row. Thus, when we sum the first n integers, we see that our vector space is of dimension $\frac{n(n+1)}{2}$.

2 Linearly Independent Sets

Let V be a vector space over an arbitrary field F, and let $\lambda \in F$ be an arbitrary scalar. Let $S = \{u_1, \ldots, u_n\}$ be a given finite subset of V. Define a new subset $S' \subset V$ by

$$S' = (S \setminus \{u_2\}) \cup \{u_2 - \lambda u_1\}$$

a. Prove that if S is linearly independent, then S' is linearly independent.

By the definition of linear independence, if

$$\sum_{i=1}^{n} a_i u_i = 0$$

for scalars $a_1, a_2, \ldots, a_n$ in F, we have that $a_1 = a_2 = \ldots = a_n = 0$.

Consider $S' = (S \setminus \{u_2\}) \cup \{u_2 - \lambda u_1\}$ where λ is a scalar in F. Suppose for some scalars $b_1, b_2, \ldots, b_{n-1}$ and b' in F, we have

$$\sum_{\substack{i=1 \\ i \neq 2}}^{n} b_i u_i + b'(u_2 - \lambda u_1) = 0$$

Expanding, we have

$$b_1 u_1 + b' u_2 - \lambda b' u_1 + \sum_{i=3}^{n} b_i u_i = 0$$

which is equivalent to

$$(b_1 - \lambda b')u_1 + b' u_2 + \sum_{i=3}^{n} b_i u_i = 0$$

We have recovered the same sum from our original set S. Since S is linearly independent, the only solution to this equation is when all coefficients are zero, i.e.,

$$b_1 - \lambda b' = 0, \quad b' = 0, \quad b_3, \ldots, b_n = 0$$

Thus, we have that $b_1 = b_2 = \ldots = b_n = 0$ as $b' = 0 \implies b_1 - \lambda b' = b_1 = 0$. Therefore, S' is also linearly independent. □

b. Prove that the $\text{span}(S) = \text{span}(S')$

Suppose $x \in \text{span}(S)$. Thus, for scalars $a_i \in F$,

$$x = \sum_{i=1}^{n} a_i u_i = a_1 u_1 + a_2 u_2 + \sum_{i=3}^{n} a_i u_i$$

This is equivalent to the following where λ is some scalar

$$x = (a_1 + \lambda a_2)u_1 + a_2(u_2 - \lambda u_1) + \sum_{i=3}^{n} a_i u_i$$

We can replace $a_1 + \lambda a_2$ with some scalar $b \in F$. Then we have

$$x = b u_1 + a_2(u_2 - \lambda u_1) + \sum_{i=3}^{n} a_i u_i \implies x \in \text{span}(S')$$

Thus, $\text{span}(S) \subset \text{span}(S')$

To show containment in the reverse, suppose $x \in \text{span}(S')$. Thus, for scalars $a_i \in F$,

$$x = \sum_{\substack{i=1 \\ i \neq 2}}^{n} a_i u_i + a_2(u_2 - \lambda u_1) = 0$$

$$\Leftrightarrow x = a_1 u_1 + a_2(u_2 - \lambda u_1) + \sum_{i=3}^{n} a_i u_i$$

We can do the opposite of above and replace a_1 with $b + \lambda a_2$ where $b \in F$. Then, we have

$$x = (b + \lambda a_2)u_1 + a_2(u_2 - \lambda u_1) + \sum_{i=3}^{n} a_i u_i = bu_1 + a_2 u_2 + \sum_{i=3}^{n} a_i u_i$$

Thus, we can write x as linear combination of elements in S, showing $x \in \text{span}(S) \implies \text{span}(S') \subset \text{span}(S)$. Thus, $\text{span}(S) = \text{span}(S')$. □

3 Basis for F^n

In order to prove that $\beta = \{e_1, \ldots, e_n\}$, where e_i is an n-tuple of scalars all 0 except for a 1 at the index i, is a basis, we need to show linear independence and span.

a. Linear Independence

Consider:

$$\sum_{i=1}^{n} \alpha_i e_i = \vec{0}$$

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i e_i &= \alpha_1 \langle 1, 0, \ldots, 0 \rangle + \alpha_2 \langle 0, 1, 0, \ldots, 0 \rangle + \cdots + \alpha_n \langle 0, \ldots, 0, 1 \rangle \\ &= \langle \alpha_1, 0, \ldots, 0 \rangle + \langle 0, \alpha_2, 0, \ldots, 0 \rangle + \cdots + \langle 0, \ldots, 0, \alpha_n \rangle \\ &= \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \end{aligned}$$

Now considering $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle = \vec{0}$, we see that comparing componentwise, each a_i must be zero for all i. Thus, since $\sum_{i=1}^{n} \alpha_i e_i = \vec{0} \implies a_i = 0 \ \forall i$, we know β is linearly independent.

b. Span: WTS: $F^n = \text{span}(\beta)$

i. $F^n \subset \text{span}(\beta)$

Let $x \in F^n$, then $x = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ for scalars $a_i \in F$. Then, we have that

$$x = \alpha_i e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \implies x = \sum_{i=1}^{n} \alpha_i e_i \implies x \in \text{span}(\beta) \implies F^n \subset \text{span}(\beta)$$

ii. $\text{span}(\beta) \subset F^n$

We could let $x \in \text{span}(\beta)$ and reverse engineer the above proof. However, we know that the span of an subset within a vector space is within the larger vector space because the vector space is closed under linear combination. Thus, $\text{span}\beta \subset F^n$. This then implies that $F^n = \text{span}(\beta)$. Therefore, β is a basis. □

4 Direct Sums

a. F^3 is the direct sum of the following

$$U_1 = \left\{(a_1, a_2, a_3) \in F^3 \mid a_2 = a_3 = 0\right\} \qquad U_2 = \left\{(a_1, a_2, a_3) \in F^3 \mid a_1 = 0\right\}$$

Consider $U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}$

We see that only the first component of U_1 can vary and only the last two components of U_2 vary. These constraints are unrelated so when added, they will disappear leaving

$$U_1 + U_2 = \left\{(a_1, a_2, a_3) \in F^3 \mid a_1, a_2, a_3 \in F\right\}$$

This is the definition of F^3 and thus is the sum.

Note that $U_1 \cap U_2$ only contains the zero vector as combining all the constraints leaves $a_1, a_2, a_3 = 0$. Thus $F^3 = U_1 \oplus U_2$.

b. Let V be a vector space over a field F, and let U_1 and U_2 be subspaces of V. We want to prove that $V = U_1 \oplus U_2$ if and only if every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$.

$(\implies)$

Given that $V = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}$, we by definition have that there exists a $u_1 \in U_1$ and $u_2 \in U_2$ such that their sum is an arbitrary vector in V.

For uniqueness, we can suppose that $v = u_1 + u_2$ and $v = u_1' + u_2'$ where $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$

Thus, we have

$$\begin{aligned} u_1 + u_2 &= u_1' + u_2' \\ u_1 - u_1' &= u_2' - u_2 \\ \implies u_1 &= u_1' \text{ and } u_2 = u_2' \end{aligned}$$

Thus, we have a unique sum for each vector in V.

$(\Leftarrow)$

Again, by definition, if we are given that every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$, V is the sum of U_1 and U_2 by definition. To show that their intersection is the zero vector, suppose $x \in U_1 \cap U_2$. This implies that $-x \in U_1 \cap U_2$ because the intersection is by definition as subspace. Then, by the vector space axioms, $x + (-x) = 0$. Since the sum is unique, x must be equal to $\vec{0}$.

5 Span and Image

Question: Let $T : V \to W$ be a linear transformation. Let $S \subset V$ be such that $\text{span}(S) = V$. Prove that

$$\text{Im}(T) = \text{span}\{T(s) \mid s \in S\}$$

Pf: Consider $y \in \text{span}\{T(s) \mid s \in S\}$

$$\begin{aligned} y &= \sum_{i=1}^{n} \alpha_i T(s_i) \\ &= T\left(\sum_{i=1}^{n} \alpha_i s_i\right) \qquad \text{def. of linear map} \end{aligned}$$

Thus, there exists an $x = \sum_{i=1}^{n} \alpha_i s_i$ such that $T(x) = y$. This implies that $x \in \text{Im}(t)$. Thus, we have shown one side of our inclusion: $\text{span}\{T(s) \mid s \in S\} \subset \text{Im}(T)$.

If $y \in \text{Im}(T)$, there exists an $x \in V$ such that $T(x) = y$ where $y \in W$. By definition, this x is some linear combination of elements in S as this set spans V. Consider

$$\begin{aligned} x &= \sum_{i=1}^{n} \alpha_i s_i \\ T(x) &= T\left(\sum_{i=1}^{n} \alpha_i s_i\right) \end{aligned}$$

$$y = T\left(\sum_{i=1}^{n} \alpha_i s_i\right)$$

$$y = \sum_{i=1}^{n} \alpha_i T(s_i) \qquad \text{def. of linear map}$$

This arbitrary linear combination shows implies that $y \in \operatorname{span}\{T(s) \mid s \in S\}$. This implies that $\operatorname{Im}(T) \subset \operatorname{span}\{T(s) \mid s \in S\}$. Because we have proven inclusion both ways, we have that $\operatorname{Im}(T) = \operatorname{span}\{T(s) \mid s \in S\}$.
□

b. Why the image of the linear map is a subspace of the codomain

Because the span of a subset of a vector space is always a subspace and we have now defined the image in terms of the span of a subset, the image of a linear map must be a subspace of the codomain.

6 Induction Proof

a.Proof

WTS:

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Base Case:

$$1 = \sum_{j=1}^{1} j^3 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1$$

Inductive Case: We want to show that $P_i \implies P_{i+1}$

Given that $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$. Consider

$$\begin{aligned}\sum_{j=1}^{n+1} j^3 &= \sum_{j=1}^{n} j^3 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \qquad \text{by inductive hypothesis} \\ &= (n+1)^2 \frac{n^2+4n+4}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4}\end{aligned}$$

Thus, we have that $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$. □

b. Connect to discussion

In discussion we showed that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$. Thus, we clearly have that $\left(\sum_{j=1}^{n} j\right)^2 = \sum_{k=1}^{n} k^3$.

7 Finite consequences of the replacement theorem

The Replacement Theorem:

Let V be a vector space over F. If G and L are subsets of V such that the span of G is V and L is linearly independent, then:

1. The number of elements in L ($\#L$) is less than or equal to the number of elements in G ($\#G$).

2. There exists a subset $H \subset V$ such that the number of elements in H (denoted $\#H$) is equal to $\#G - \#L$, and the span of $H \cup L$ is V.

We define $L = \{v_1, \ldots, v_m\}$ and $G = \{u_1, \ldots, u_n\}$.

a. Number of elements in a spanning set

We want to show that every spanning set $G \subset V$ contains at least n elements. Let's suppose that $G \subset V$ is an arbitrary spanning set. We can say that there exists an $L \subset V$ such that L is linearly independent. Then, by the replacement theorem, there exists an $H \subset V$ such that $\text{span}(H \cup U) = V$. Furthermore, we have that

$$\#H = \#G - \#L \implies \#G = \#H + \#L$$

We already know that $\text{span}(H \cup U) = V$. $H \cup U$ has at most $\#H + \#L$ elements. Thus, an arbitrary spanning set has at least as many elements as L, or at least n elements. $\square$

b. Span and basis

Assume S is L.D. WLOG, this implies that

$$v_1 = \sum_{i=2}^{n} \alpha_i v_i \implies \text{span}(S) = \text{span}\{v_2, ...v_n\}$$

This is a contradiction because this is a spanning set with $n-1$ elements which is not possible given our last result. Thus, because S spans the vector space and is linearly independent, we have that S is a basis. $\square$

c. Every spanning set contains a basis

Suppose $G = \{u_1, u_2, \ldots, u_n, \ldots, u_m\}$ is a spanning set for V over a field F with m elements–which we know has at least n elements, where n is the number of elements in the maximal linearly independent set from (a). While $\#G > n$, we can write one element (WLOG we will say it is the last element in the set). We can first write the following where $a_i \in F \, \forall i$

$$u_m = \sum_{i=1}^{m-1} a_i u_i$$

Then, we can remove that element from the set G. We can iterate this process because we know by the replacement theorem and the definition of a bases that G will not be linearly independent if it has more than n elements. Each time we remove an element, it will not change the span because the vector we removed was in the span of the remaining vectors. Thus, we can iterate until our basis–meaning it is linearly independent and spans is

$$\{u_1, u_2, \ldots, u_n\}$$

Therefore, an arbitrary spanning set G contains a basis. $\square$

8 Infinite consequences of the replacement theorem

Given that S is a linearly independent subset of V, by the replacement theorem, if there exists another subset $G \subset V$ such that G spans V, G must have the same or more elements than S. This directly implies that any spanning set is infinite as well. $\square$

9 Subspaces and Dimension

Suppose that V is a finite-dimensional vector space and $U \subset V$ is a subspace.

a. We want to prove that the $\dim(U)$ is finite and less than or equal to that of V.

Because V is finite, it has some basis $\beta = \{u_1, u_2, \ldots, u_n\}$. Because $U \subset V$, if we suppose $x \in U$, we have that $x \in V$ and can thus express $x = \sum_{i=1}^{n} \alpha_i u_i$, which implies that an arbitrary element in U is in the span of a finite set of vectors, implying that any bases must have fewer or the same number of vectors as U from the replacement theorem. $\square$

b. We want to show that if also $\dim(U) = \dim(V) = n$, then $U = V$.

Since U is subspace, it has its own basis, with n elements. By definition, this basis is linearly independent and spans U. However, V also has a basis with n elements. Since, the basis for U is linearly independent and has n elements, it must also be a basis for V by the replacement theorem. If two vector spaces have the same basis, they are equal. Therefore, $U = V$. $\square$

10 Subspace and Dimension Textbook Proofs

Let W_1 and W_2 be subspaces of a vector space V with dimensions m and n respectively where $m \geq n$.

a. We want to show that $\dim(W_1 \cap W_2) \leq n$.

Because the intersection of subspaces is a subspace, we can suppose $W_1 \cap W_2$ has a basis $\beta = \{u_1, u_2, \ldots, u_k\}$. We know that W_2 has a basis $\beta_2 = \{u_1, u_2, \ldots, u_n\}$. If we suppose that $x \in (W_1 \cap W_2)$, we can write $x = \sum_{i=1}^{k} \alpha_i u_i$ where $\alpha_i \in F \, \forall i$. By the definition of intersection, we know $x \in W_2$. This implies that an arbitrary linearly combination of the elements in a basis for $W_1 \cap W_2$ is in W_2, which means that W_2 must have an equal or greater number of basis vectors as $W_1 \cap W_2$. Thus, $\dim(W_1 \cap W_2) \leq n$. $\square$

b. We want to show that $\dim(W_1 + W_2) \leq m + n$.

We can have a spanning set for $W_1 + W_2$ to be the union of their bases because the sum of subspaces boils down to all pairwise combinations of their elements. This can also be written as the linear combinations of all their individual basis vectors. The number of elements in the basis for $\#\beta_1 = m$ and $\#\beta_2 = n$. This implies that the union of these bases will have a maximum of $m + n$ elements by the definition of union. From the replacement theorem and its consequences, a basis will have less than or equal to the number of elements in a spanning set. Thus, $W_1 + W_2$ will have a basis with a maximum of $m + n$ elements. Therefore, $\dim(W_1 + W_2) \leq m + n$. $\square$

11 Direct Sum Proof

If we have that W_1 and W_2 are subspaces of a vector space V such that $V = W_1 \oplus W_2$ and β_1 and β_2 are the bases for W_1, W_2 respectively. We want to show that: i. $\beta_1 \cap \beta_2 = \phi$ and ii. $\beta_1 \cup \beta_2$ is a basis for V.

i. Suppose $\beta_1 \cap \beta_2 \neq \phi$. This would imply that both W_1, W_2 contain some non-zero vector. However, by the definition of direct sum, their intersection is only the zero vector. Thus, by contradiction, $\beta_1 \cap \beta_2 = \phi$. $\square$

ii. We want to show linear independence and span. Note that the union $\beta_1 \cup \beta_2$ will simply have the sum of the number of vectors of each since we just showed that their intersection is empty. Consider the following where $a_i, b_i \in F$ and β_1, β_2 are made up of a collection of vectors u_i, v_i respectively.

$$\sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{m} b_i v_i = \vec{0}$$

Because there is no intersection between the bases, this is equivalent to

$$\sum_{i=1}^{n} a_i u_i = \vec{0} \qquad \text{and} \qquad \sum_{i=1}^{m} b_i v_i = \vec{0}$$

Both of these statements imply that $a_i = 0 \, \forall i$ and $b_i = 0 \, \forall i$ because both bases are linearly independent on their own.

To show span, we know that for $z \in V$ there exists a $x \in W_1$ and a $y \in W_2$ such that $z = x + y$. We can represent x as a linear combination of the first n basis vectors and y as a linear combination of the last m. In each case the rest of the scalars of the linear combination of the full basis can be set to 0. This then implies that z can be represented as a sum of these linear combinations, which could just be combined into one linear combination. Thus, an arbitrary vector in V can be represented as a linear combination of $\beta_1 \cup \beta_2$, and thus $\beta_1 \cup \beta_2$ spans V. Therefore, $\beta_1 \cup \beta_2$ is a basis for V. $\square$