33B Notes

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1 Linear Algebra

Augmented Matrix
An augmented matrix for a system of m equations in n variables is a rectangular array with m rows and $n+1$ columns which stores all the coefficients of the system. It is given by

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gaussian Elimination Objectives
Suppose we are playing a game called Gaussian Elimination with the following rules:

(I) There are three legal moves (so-called elementary row operations) which we can use to transform our augmented matrix.

(II) First, get the matrix to Row Echelon Form.

(III) Next, transform the matrix into Reduced Row Echelon Form.

(IV) Read off solutions.

Elementary Row Operations
The following are the elementary row operations:

(1) **Row Switching**: A row in the matrix can be switched with another row in the matrix. $R_i \leftrightarrow R_j$.

(2) **Row Multiplication**: A row can be multiplied by a non-zero constant. $\alpha R_i \to R_i$.

(3) **Row Addition**: A row can be replaced with the sum of that row and a multiple of another row. $R_i + \alpha R_j \to R_i$.

Note: Row operations are limited by the required property that they be reversible.

Row Echelon Form
An augmented matrix is in row echelon form (REF) if

1. Every row with nonzero entries is above every row with all zeroes (if there are any).

2. The leading coefficient (i.e., the leftmost nonzero entry) of a nonzero row is to the right of the leading coefficient of the row above it.

*Note: Given an augmented matrix in REF, a **pivot** is a leading coefficient in a nonzero row.*

Reduced Row Echelon Form

An augmented matrix is in reduced row echelon form (RREF) if

1. It is in row echelon form (REF).
2. Every pivot (the leading coefficient in a nonzero row) is 1.
3. Every entry above a pivot is 0.

Pivot and Free Variables

$$\begin{array}{cccc|c} X_1 & X_2 & X_i & X_n & \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Pivot variables are variables X_i such that the column they are associated with has a pivot. Free variables are variables whose columns do have a pivot. Free variables take arbitrary values and can be represented in the following way

$$X_i = s \in \mathbb{R}$$

Then, the pivot variables can be written in terms of the free variables.

Solution of a System of Linear Equations

Given a system of equations, exactly one of the following three things will happen:

1. The system has zero solutions (i.e., it is inconsistent). This happens when the RREF contains a row of the form
$$[0 \cdots 0 \,|\, 1]$$
because this corresponds to the equation $0 = 1$ which can never be true.
2. The system has exactly one solution. This happens when the system is consistent and there are no free variables in the RREF.
3. The system has infinitely many solutions. This happens when the system is consistent and there is at least one free variable in the RREF.

Rank

We define the rank of a matrix to be the number of pivots any REF of that matrix has (it will be the same number even though there could be many different REFs).

The system is inconsistent if the rank of the coefficient matrix is not equal to the rank of the entire augmented matrix. The system has exactly one solution if the rank of the augmented matrix is equal to the number of columns in the coefficient matrix. The system has infinitely many solutions if the rank of the augmented matrix is less than the number of columns in the coefficient matrix.

Nice Set

We call a set $D \subseteq \mathbb{R}$ nice if it is an interval or a union of a sequence of intervals, i.e., if there exists a sequence of intervals $I_0, I_1, I_2, \ldots$ such that

$$D = \bigcup_{n \geq 0} I_n$$

2 First-order Differential Equations

Implicit Differential Equation of Order r
An implicit differential equation (of order r) is an equation which can be written in the form

$$F(t, y, y', y'', \ldots, y^{(r)}) = 0$$

where F is a real-valued function of $r+2$ variables. The order is the order r of the highest derivative $y^{(r)}$ of y which appears in the equation.

A solution to the above equation is a function $y : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) which is differentiable at least r times such that

$$F(t, y(t), y'(t), \ldots, y^{(r)}(t)) = 0$$

for every $t \in I$, i.e., for every $t \in I$, when you plug $t, y(t), y'(t), \ldots, y^{(r)}(t)$ into the function F the output is zero.

Normal Form of a Differential Equation
A differential equation of order r in normal form (or an explicit differential equation of order r) is a differential equation which can be written in the form

$$y^{(r)} = F(t, y, y', y'', \ldots, y^{(r-1)})$$

where F is a real-valued function of $r+1$ variables. A solution of the above equation is a function $y : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) which is at least r times differentiable, such that for every $t \in I$:

$$y^{(r)}(t) = F(t, y(t), y'(t), \ldots, y^{(r-1)}(t))$$

Thus, an explicit first-order differential equation would take the form

$$y' = F(t, y)$$

First Order Linear Differential Equation Form
A first-order linear differential equation is a differential equation which can be written in the form:

$$y' + f(t)y = g(t)$$

where f and g are real-valued functions of the variable t. The functions $f(t)$ and $g(t)$ are called the coefficient functions. This has the following subcategories, specialized beyond the general form

1. Direct Integration: This is the case when $f(t) = 0$ for all t. The solution may be achieved by simply integrating both sides with respect to t. It has the following form

   $$y' = g(t)$$

2. Homogeneous: This is the case when $g(t) = 0$ for all t and $f(t)$ is potentially non-zero. It has the following form

   $$y' + f(t)y = 0$$

   - This requires multiplication by an integrating factor $\mu(t) := \exp\left(\int f(t)\,dt\right)$
   - The solution takes the form $y(t) = y(t; C) = \frac{C}{\mu(t)} = C\exp\left(-\int f(t)\,dt\right)$

Solution to General Form of a First Order Linear Equation
Suppose $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are continuous functions with nice domains $D, E \subseteq \mathbb{R}$ and consider the differential equation:

$$y' + f(t)y = g(t)$$

1. Define the integrating factor to be the function $\mu : D \to \mathbb{R}$ given by:

$$\mu(t) := \exp\left(\int f(t)\, dt\right)$$

   (here $\int f(t)\, dt$ can be any antiderivative of $f(t)$, the constant of integration does not matter). Then we can multiply our original equation by μ to obtain:

$$\mu(t)(y' + f(t)y) = (\mu(t)y)' = \mu(t)g(t)$$

2. The general solution of $y' + f(t)y = g(t)$ is then given by:

$$y(t) = y(t; C) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\, dt + C\right)$$

   Furthermore, suppose we are also given an initial condition $y(t_0) = y_0$, where $t_0 \in D \cap E$ and $y_0 \in \mathbb{R}$.

3. Then the initial value problem has the unique solution:

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\, ds + y_0\right)$$

   where $\mu(t) := \exp\left(\int_{t_0}^{t} f(s)\, ds\right)$.

4. The interval of existence of this solution is the largest interval $I \subseteq \mathbb{R}$ such that:
   (a) $t_0 \in I$,
   (b) $I \subseteq D$, and
   (c) $I \subseteq E$.

Proof of Key Property of the Integrating Factor $\mu(t)$.

$$\begin{aligned}
(\mu(t)y)' &= \frac{d}{dt}(\mu(t)y) \\
&= \frac{d}{dt}\left(\exp\left(\int f(t)\, dt\right) \cdot y\right) \\
&= \exp\left(\int f(t)\, dt\right) \cdot \frac{d}{dt}(y) + y \cdot \frac{d}{dt}\left(\exp\left(\int f(t)\, dt\right)\right) \\
&= \mu(t)y' + y \cdot \exp\left(\int f(t)\, dt\right) \cdot f(t) \\
&= \mu(t)y' + f(t) \cdot \exp\left(\int f(t)\, dt\right) \cdot y \\
&= \mu(t)y' + f(t)\mu(t)y
\end{aligned}$$

3 First Order Non-linear Differential Equations

Differential Form
A differential form is a formal expression of the form:

$$P(t, y)\, dt + Q(t, y)\, dy$$

where P, Q are functions of two variables and dt and dy are meaningless placeholders associated to the variables t and y, called differentials. Differential forms can be added together in the natural way, and you can multiply them (from the left) by arbitrary functions $R(t, y)$.

Differential
Given a two-variable function $F(t, y)$, the differential of F (notation: dF) is the differential form:

$$dF := \frac{\partial F}{\partial t}(t, y)\, dt + \frac{\partial F}{\partial y}(t, y)\, dy$$

Exact Differential Equation to Differential Form Equation
Exact differential equations are represented in the general form $y' = f(t, y)$, where f is a two-variable function which might not be separable, i.e., it might not be of the form $f(t, y) = g(t)h(y)$. To convert this into a differential form equation, follow these steps:

1. Rewrite as $\frac{dy}{dt} = f(t, y)$.
2. "Multiply" both sides by dt, then add $-f(t, y)dt$ to both sides to obtain:

$$-f(t, y)dt + dy = 0.$$

3. Multiply both sides by a carefully chosen integrating factor $\mu(t, y)$:

$$-f(t, y)\mu(t, y)dt + \mu(t, y)dy = 0.$$

This process results in a differential form equation: $P(t, y)dt + Q(t, y)dy = 0$, where $P(t, y) = -f(t, y)\mu(t, y)$ and $Q(t, y) = \mu(t, y)$.

Potential Functions
A potential function for a differential form equation is a two-variable function $F(t, y)$ such that

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial y}dy = P(t, y)dt + Q(t, y)dy,$$

i.e.,

1. $\frac{\partial F}{\partial t} = P(t, y)$, and
2. $\frac{\partial F}{\partial y} = Q(t, y)$.

In other words, a potential function is like an antiderivative of a differential form. Unfortunately, not every differential form has a potential function.

Exact Differential Forms and Necessary Condition

Suppose $P, Q : D \to \mathbb{R}$ are continuous two-variable functions on a nice domain $D \subseteq \mathbb{R}^2$, and are also continuously differentiable. A differential form

$$P\,dt + Q\,dy$$

is exact if there exists a continuously differentiable function $F : D \to \mathbb{R}$ such that

$$dF = P\,dt + Q\,dy.$$

Closed

Suppose $P, Q : D \to \mathbb{R}$ are continuously differentiable two-variable functions on a nice domain $D \subseteq \mathbb{R}^2$. We say that the differential form

$$P\,dt + Q\,dy$$

is closed if

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial t} = 0,$$

i.e., if the left-hand side is the constant zero function. This condition ensures that the changes in P with respect to y and Q with respect to t balance each other out, indicating a kind of symmetry in the differential form's structure.

Converse of Exact $\implies$ Closed

Suppose $P, Q : I \times J \to \mathbb{R}$ are continuously differentiable functions and $I, J \subseteq \mathbb{R}$ are intervals (so the common domain of P and Q is a rectangle). Then the following are equivalent:

1. The differential form $P\,dt + Q\,dy$ is exact.
2. The differential form $P\,dt + Q\,dy$ is closed.

A differential form is *closed* if its coefficients satisfy the condition $\frac{\partial Q}{\partial t} = \frac{\partial P}{\partial y}$. This condition ensures that local differentiability properties imply the existence of a potential function, making the form exact.

Finding a Potential Function of an <u>Exact</u> Differential Form Equation

The solution to the differential form equation $P(t,y)\,dt + Q(t,y)\,dy = 0$ is $F(t,y) = C$, where F is a potential function of $P(t,y)\,dt + Q(t,y)\,dy$.

1. First solve $\frac{\partial F}{\partial t} = P$ by integrating with respect to t:

$$F(t,y) = \int P(t,y)\,dt + \phi(y)$$

   where $\phi(y)$ acts as a constant but is a function of y.

2. Next, we need to find what $\phi(y)$ is. Since we know $\frac{\partial F}{\partial y} = Q(t,y)$, we differentiate our previous result with respect to y:

$$\frac{\partial}{\partial y}\left(\int P(t,y)\,dt\right) + \phi'(y) = Q(t,y)$$

   , and thus

$$\phi(y) = \int \left(Q(t,y) - \frac{\partial}{\partial y}\int P(t,y)\,dt\right)\,dy$$

3. Put it all together in the following form:

$$F(t,y) = C$$

Example:

$$(2t \sin y + y^3 e^t)\, dt + (t^2 \cos y + 3y^2 e^t)\, dy = 0.$$

$$\begin{aligned}
\frac{\partial}{\partial y}(2t \sin y + y^3 e^t) &= 2t \cos y + 3y^2 e^t, \\
\frac{\partial}{\partial t}(t^2 \cos y + 3y^2 e^t) &= 2t \cos y + 3y^2 e^t, \\
&\implies \text{Exact} \\
F(t, y) = \int (2t \sin y + y^3 e^t)\, dt + \phi(y) &= t^2 \sin y + y^3 e^t + \phi(y), \\
\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(t^2 \sin y + y^3 e^t + \phi(y)) = t^2 \cos y + 3y^2 e^t + \phi'(y) &= t^2 \cos y + 3y^2 e^t, \\
\phi'(y) &= 0, \\
\phi(y) &= C, \\
F(t, y) &= t^2 \sin y + y^3 e^t + C, \\
t^2 \sin y + y^3 e^t + C &= 0, \\
t^2 \sin y + y^3 e^t &= C.
\end{aligned}$$

Integrating Factor to Make Differential Form Exact

Suppose $P, Q : D \to \mathbb{R}$ are continuous on a nice domain $D \subseteq \mathbb{R}^2$. We say that a function $\mu : D \to \mathbb{R}$ is an integrating factor for the differential form equation

$$P(t, y)\, dt + Q(t, y)\, dy = 0$$

if

(i) $\mu(t, y) \neq 0$ for every $(t, y) \in D$, and

(ii) $\mu(t, y)P(t, y)\, dt + \mu(t, y)Q(t, y)\, dy$ is exact.

Existence Theorem

Suppose $f : I \times J \to \mathbb{R}$ is a continuous two-variable function defined on a rectangle $I \times J$ in the ty-plane, where $I, J \subseteq \mathbb{R}$ are intervals. Given any point $(t_0, y_0) \in I \times J$, the initial value problem

1. $y' = f(t, y)$
2. $y(t_0) = y_0$

has a solution $y(t)$ defined on some interval $I' \subseteq I$ that contains t_0. Furthermore, the solution will be defined at least until the solution curve $t \mapsto (t, y(t))$ leaves the rectangle $I \times J$.

Uniqueness Theorem

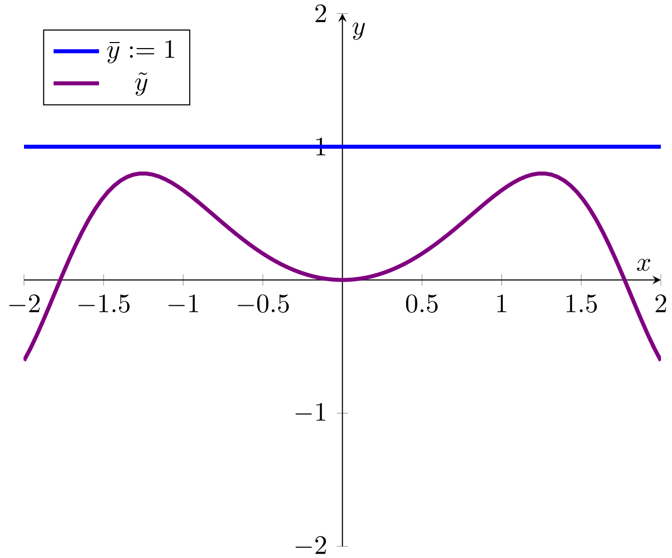
Suppose $f : I \times J \to \mathbb{R}$ is a continuous two-variable function defined on a rectangle $I \times J$ in the ty-plane, and $I, J \subseteq \mathbb{R}$ are intervals. Additionally, suppose the partial derivative $\frac{\partial f}{\partial y}$ exists and is continuous on all of $I \times J$. Given $(t_0, y_0) \in I \times J$, and assuming two solutions $y(t)$ and $\tilde{y}(t)$ to the same initial value problem (IVP):

1. $y'(t) = f(t, y(t))$ and $\tilde{y}'(t) = f(t, \tilde{y}(t))$ for every t,
2. $y(t_0) = y_0$ and $\tilde{y}(t_0) = y_0$,

then for every t such that $(t, y(t))$ and $(t, \tilde{y}(t))$ remain within the rectangle $I \times J$, it follows that $y(t) = \tilde{y}(t)$.

Practical Benefit: Given the conditions of this theorem, different solution curves cannot cross.

Here is an example of an application: Suppose $\bar{y}$ is a solution to a differential equation and $\tilde{y}$ is a solution as well that passes thru the origin. Then $\tilde{y} < 1 \quad \forall t \in I$



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4 Autonomous Equations

Autonomous Equations

A first-order differential equation is called an autonomous equation if it can be written in the form:

$$y' = f(y)$$

i.e., if the equation does not depend on the independent variable t.

Remarks:

1. The direction field does not change as you go from left to right, it only changes as you go from bottom to top. This is because the function $f(t, y) = f(y)$ is only a function of y and does not depend on t.

2. Suppose $y_0(t)$ is a particular solution and $C \in \mathbb{R}$ is a constant. Then $y_0(t+C)$ (a shift of y_0 to the left by C) is also a solution. Indeed:

$$(y_0(t+C))' = y_0'(t+C) = f(y_0(t+C))$$

3. Suppose $y_0 \in \mathbb{R}$ is such that $f(y_0) = 0$. Then the constant function $y(t) := y_0$ for all t is a solution to $y' = f(y)$. Such a number y_0 is called an equilibrium point and the constant function $y(t) := y_0$ is called an equilibrium solution.

Phase Line

A phase line for the equation $y' = f(y)$ is a plot of the y-axis (displayed horizontally) with the following features:

1. At every equilibrium point y_0 (i.e., where $f(y_0) = 0$), there is a dot.

2. In a region between two equilibrium points (or between an equilibrium point and $\pm\infty$), if $f(y) < 0$ in that region, then there is an arrow to the left. This tells us that for these y-values, the solution is strictly decreasing.

3. In a region where $f(y) > 0$, then there is an arrow to the right. This tells us that for these y-values, the solution is strictly increasing.

4. At each equilibrium point y_0, if the two arrows on either side of y_0 are both pointing towards y_0, then the dot at y_0 is filled in. Otherwise, the dot is not filled in.

Often the phase line is plotted with a vertical $f(y)$-axis as well, superimposed with a graph of the function $f(y)$.

Stability
Consider the autonomous equation $y' = f(y)$. Suppose $y_0 \in \mathbb{R}$ is an equilibrium point (i.e., $f(y_0) = 0$). We say that y_0 is

1. asymptotically stable if a solution which goes through a point $(t_0, y_0 + \epsilon)$, where $|\epsilon| \ll 1$ is very tiny, will asymptotically approach the solution $y(t) = y_0$. These correspond to the filled-in dots on the phase line.

2. unstable if it is not asymptotically stable, i.e., if there is some solution which goes through a point $(t_0, y_0 + \epsilon)$ which "peels off" and is not asymptotic to the solution $y(t) = y_0$. These correspond to the non-filled-in dots on the phase line.

In other words, asymptotically stable equilibrium points act like "sinks", bringing nearby solution curves towards the constant solution at that point. Unstable equilibrium points, at least on one of the two sides, will "repel" nearby solution curves.

$\Rightarrow$ ***First Derivative Test for Stability***

(1) if $f'(y_0) < 0$, then f is strictly decreasing at y_0 and y_0 is asymptotically stable,

(2) if $f'(y_0) > 0$, then f is strictly increasing at y_0 and y_0 is unstable,

(3) if $f'(y_0) = 0$, then no conclusion can be drawn and further investigation is needed.

Note:

1. By studying the function $f(y)$, first construct the phase line, including classifying the equilibrium points as either asymptotically stable or unstable,

2. In the direction field, plot the equilibrium solutions.

3. In the other regions, plot solution curves that behave according to the phase line: if the phase line points to the left, the solution should be strictly decreasing and asymptotic to the next lower equilibrium solution (or diverge to $-\infty$). If the phase line points to the right, the solution should be strictly increasing and asymptotic to the next higher equilibrium solution (or diverge to $+\infty$).

Basically, filled in dot $\implies$ stable, empty dot $\implies$ unstable.

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5 Second-Order Linear Differential Equations

Second-Order Linear Differential Equation
A second-order linear differential equation is a differential equation which can be put in the form:

$$y''(t) + p(t)y' + q(t)y = g(t)$$

where the coefficient functions p, q, and g are functions of the independent variable t only. The function $g(t)$ is referred to as the forcing term. If $g(t) = 0$ is the constant zero function, then the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

is said to be homogeneous.

Second Order Linear Existence and Uniqueness Theorem
Suppose $p, q, g : I \to \mathbb{R}$ are continuous functions with domain $I \subseteq \mathbb{R}$ being an interval. Then, given $t_0 \in I$ and any two real numbers $y_0, y_1 \in \mathbb{R}$, there is a unique function $y : I \to \mathbb{R}$ which satisfies the initial value problem:

(i) $y'' + p(t)y' + q(t)y = g(t)$

(ii) $y(t_0) = y_0$ and $y'(t_0) = y_1$.

For homogeneous linear equations, given two solutions, we can mass-produce many more solutions.

Solution to Homogeneous Second Order Linear Differential Equations
Suppose y_1, y_2 are linearly independent solutions to the homogeneous second-order linear equation

$$y'' + p(t)y' + q(t)y = 0$$

Then the general solution is:

$$y(t; C_1, C_2) = C_1 y_1(t) + C_2 y_2(t)$$

The pair, $\{y_1, y_2\}$ is called a **fundamental set of solutions** to the homogeneous second-order linear equation.

Proof. Let $C_1, C_2 \in \mathbb{R}$ be arbitrary. Note that

$$\begin{aligned}
&(C_1y_1 + C_2y_2)'' + p(t)(C_1y_1 + C_2y_2)' + q(t)(C_1y_1 + C_2y_2)\\
&= (C_1y_1'' + C_2y_2'') + p(t)(C_1y_1' + C_2y_2') + q(t)(C_1y_1 + C_2y_2) \quad \text{(because the derivative is linear)}\\
&= C_1(y_1'' + p(t)y_1' + q(t)y_1) + C_2(y_2'' + p(t)y_2' + q(t)y_2)\\
&= C_1 \cdot 0 + C_2 \cdot 0\\
&= 0,
\end{aligned}$$

because y_1 and y_2 are both solutions. Thus, $C_1y_1 + C_2y_2$ is also a solution. □

Note: This math works for the first-order homogeous linear differential equation as well. However, it would be impossible to find two linearly independent solutions. We are essentially finding the basis of our solution space and thus will only find n basis vectors in an nth-order differential equation. In linear algebra terms, a "fundamental set of solutions" is a basis of the subspace of all solutions.

Linearly Independent Functions
Suppose $y_1, y_2 : I \to \mathbb{R}$ are functions defined on an interval $I \subseteq \mathbb{R}$. We say that y_1 and y_2 are linearly independent if, for every $C_1, C_2 \in \mathbb{R}$, if

$$C_1y_1(t) + C_2y_2(t) = 0 \text{ for every } t \in I,$$

then $C_1 = C_2 = 0$. In other words, y_1 and y_2 are linearly independent if the only way for a linear combination of y_1 and y_2 to be the constant zero function is with the trivial linear combination $0y_1 + 0y_2$. If y_1 and y_2 are not linearly independent, then we say they are linearly dependent.

For two functions y_1 and y_2 to be linearly dependent, this means that either y_1 is a constant multiple of y_2 (i.e., $y_1 = Cy_2$ for some $C \in \mathbb{R}$) or y_2 is a constant multiple of y_1 ($y_2 = Cy_1$ for some $C \in \mathbb{R}$).

Wronskian

Suppose $u, v : I \to \mathbb{R}$ are two differentiable functions defined on an interval $I \subseteq \mathbb{R}$. Define the Wronskian of u and v to be the function $W : I \to \mathbb{R}$ defined by

$$W(t) := \det \begin{bmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{bmatrix} := u(t)v'(t) - v(t)u'(t)$$

for all $t \in I$.

Wronskian Dichotomy I

Suppose $p, q, u, v : I \to \mathbb{R}$ are functions defined on an interval $I \subseteq \mathbb{R}$ such that u and v are solutions to the homogeneous second-order linear differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Let $W(t)$ be the Wronskian of u and v. Then exactly one of the following two things is true:

(Case 1) $W(t) = 0$ for all $t \in I$, or

(Case 2) $W(t) \neq 0$ for all $t \in I$.

Proof. We are assuming that both u and v satisfy:

$$u'' + pu' + qu = 0 \text{ and } v'' + pv' + qv = 0.$$

We wish to show that $W = uv' - vu'$ is either everywhere zero or everywhere nonzero. First, differentiate W:

$$W' = uv'' + u'v' - vu'' - v'u' = uv'' - vu'' = u(-pv' - qv) - v(-pu' - qu)$$

because u, v are solutions, leading to

$$W' = -puv' - quv + pvu' + quv = -p(uv' - vu') = -pW.$$

Thus, the function $W(t)$ is a solution to the first-order linear homogeneous equation $W' + pW = 0$. Pick t_0 in the domain of W, and suppose $W(t_0) = W_0$. Then by the existence and uniqueness theorem, we have that

$$W(t) = W_0 \exp\left(-\int_{t_0}^{t} p(s)\, ds\right).$$

Thus, if $W_0 = 0$, we are in Case 1. Otherwise, if $W_0 \neq 0$, we are in Case 2, since the exponential function is never zero.

□

Wronskian Dichotomy II

Let $p, q, u, v : I \to \mathbb{R}$ be functions defined on an interval $I \subseteq \mathbb{R}$, with u and v as solutions to the homogeneous second-order linear differential equation $y'' + p(t)y' + q(t)y = 0$. Define $W(t)$, the Wronskian of u and v, as follows. Then, the relationship between $W(t)$ and the linear (in)dependence of u and v is characterized by two cases:

1. If there exists some $t_0 \in I$ for which $W(t_0) = 0$, implying that $W(t) = 0$ for all $t \in I$, then u and v are linearly dependent.

2. Conversely, if there exists some $t_0 \in I$ for which $W(t_0) \neq 0$, implying that $W(t) \neq 0$ for all $t \in I$, then u and v are linearly independent.

6 Homogeneous Second Order Linear Differential Equations with Constant Coefficients

Characteristic Polynomial
The characteristic polynomial associated with the homogeneous second-order linear equation

$$y'' + py' + qy = 0$$

(where $p, q \in \mathbb{R}$ are constant functions) is the quadratic polynomial

$$f(\lambda) = \lambda^2 + p\lambda + q$$

in the variable λ. A root of the characteristic polynomial is called a characteristic root.

Solutions to Homogeneous Second Order Linear Differential Equations with Constant Coefficients
We have three cases using the roots of our characteristic polynomial associated with the differential equation $y'' + py' + qy = 0$:

Case I: Distinct real roots. The general solution to $y'' + py' + qy = 0$ when $\lambda_1 \neq \lambda_2$ are distinct and real is:

$$y(t; C_1, C_2) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Case II: Repeated real roots. The general solution to $y'' + py' + qy = 0$ when $\lambda_1 = \lambda_2$ are not distinct (and real) is:

$$y(t; C_1, C_2) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$

Case III: Distinct complex roots, complex version. The general solution to $y'' + py' + qy = 0$ when $\lambda_1 = a + bi$, $\lambda_2 = a - bi$ are distinct and complex is:

$$y(t; C_1, C_2) = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Case III: Distinct complex roots, real version. The general solution to $y'' + py' + qy = 0$ when $\lambda_1 = a + bi$, $\lambda_2 = a - bi$ are distinct and complex is:

$$y(t; C_1, C_2) = C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt)$$

Properties of Complex Numbers
Recall the following properties about complex numbers:

1. A complex number is a number of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$ is the imaginary unit. We denote the set of all complex numbers by $\mathbb{C}$.

2. Given a complex number $z = a + bi$, we define its real part to be $Re(z) := a$ and its imaginary part to be $Im(z) := b$.

3. Given a complex number $z = a + bi$, we define its complex conjugate to be $\bar{z} := a - bi$.

4. Here are some facts about the complex conjugate of a complex number $z = a + bi$:

(a) $\bar{\bar{z}} = z$
(b) $Re(z) = (z + \bar{z})/2$
(c) $Im(z) = (z - \bar{z})/(2i)$
(d) $z = \bar{z}$ iff $z \in \mathbb{R}$ iff $b = 0$.
(e) For $w \in \mathbb{C}$ we have $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z} \cdot \bar{w}$

5. The complex exponential function behaves according to Euler's formula:

$$e^{a+bi} = e^a(\cos b + i \sin b)$$

6. Suppose $f(\lambda) = \lambda^2 + p\lambda + q$ is a polynomial with real coefficients $p, q \in \mathbb{R}$ and a complex (non-real) root $\lambda_1 = a + bi$. Then $\lambda_2 := \bar{\lambda}_1 = a - bi$ is also a complex root, i.e., the complex roots of a real polynomial occur in complex conjugate pairs.

7. Suppose $z(t)$ is a complex-valued function such that $z(t) = x(t) + y(t)i$, where $x(t), y(t)$ are real-valued functions. Then

$$\frac{d}{dt}z(t) = \frac{d}{dt}x(t) + i\frac{d}{dt}y(t)$$

i.e., complex-valued functions can be differentiated by separately differentiating the real and imaginary parts in the usual way.

7 Non-homogeneous Second-Order Differential Equations

General Solutions to Inhomogeneous Differential Equations
Suppose $y_p(t)$ is a particular solution to the inhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{(A)}$$

and that $y_1(t), y_2(t)$ form a fundamental set of solutions to the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \quad \text{(B)}.$$

Then the general solution to the inhomogeneous equation (A) is given by

$$y(t) = y(t; C_1, C_2) = C_1y_1(t) + C_2y_2(t) + y_p(t).$$

Method of Undetermined Coefficients
Suppose $y'' + py' + qy = g(t)$ is an inhomogeneous differential equation such that:

(a) $p, q \in \mathbb{R}$ are constants, and

(b) $g(t)$ is not a solution to the homogeneous solution $y'' + py' + qy = 0$.

Then the following gives the trial solution you should guess depending on the form of the forcing function $g(t)$ (where $A, B, a, b, r, \omega \in \mathbb{R}$, $P(t)$ is a polynomial and $p_0(t), p_1(t)$ are polynomials of the same degree as P). If the forcing function $g(t)$ is of the form:

1. e^{rt}, then the trial solution is $y_p(t) = ae^{rt}$.

2. $A \cos \omega t + B \sin \omega t$, then the trial solution is $y_p(t) = a \cos \omega t + b \sin \omega t$.

3. $P(t)$, then the trial solution is $y_p(t) = p_0(t)$.

4. $P(t) \cos \omega t$ or $P(t) \sin \omega t$, then the trial solution is $y_p(t) = p_0(t) \cos \omega t + p_1(t) \sin \omega t$.

5. $e^{rt}\cos\omega t$ or $e^{rt}\sin\omega t$, then the trial solution is $y_p(t) = e^{rt}(a\cos\omega t + b\sin\omega t)$.

6. $e^{rt}P(t)\cos\omega t$ or $e^{rt}P(t)\sin\omega t$, then the trial solution is $y_p(t) = e^{rt}(p_0(t)\cos\omega t + p_1(t)\sin\omega t)$.

If $g(t)$ is a solution to $y'' + py' + qy$, then use the trial solution $ty_p(t)$, and if that does not work, then use the trial solution $t^2y_p(t)$.

Example:
Find a particular solution to:

$$y'' + 3y' + 2y = 4e^{-3t}.$$

Solution. Here the forcing term is $g(t) = 4e^{-3t}$. We will guess that there is a particular solution of the form $y_p(t) = ae^{-3t}$, where $a \in \mathbb{R}$ is an undetermined coefficient. Thus in this case our "trial solution" is a function $y_p(t) = ae^{-3t}$. To find a, we plug the trial solution $y_p(t)$ into the equation:

$$y_p'' + 3y_p' + 2y_p = 9ae^{-3t} - 9ae^{-3t} + 2ae^{-3t} = 4e^{-3t}.$$

This simplifies to

$$(9a - 9a + 2a)e^{-3t} = 2ae^{-3t} = 4e^{-3t}$$

and so $2a = 4$, i.e., $a = 2$. Thus the function $y_p(t) = 2e^{-3t}$ is a particular solution to $y'' + 3y' + 2y = 4e^{-3t}$.

Variation of Parameters

Suppose $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions to the homogeneous differential equation:

$$y'' + p(t)y' + q(t)y = 0,$$

where, in particular, the Wronskian $W(t) := y_1y_2' - y_2y_1' \neq 0$ for all t. Then, the inhomogeneous differential equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

has the following as a particular solution:

$$y_p(t) = y_1(t)\int \frac{-y_2(t)g(t)}{W(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\,dt.$$

Example of Variation of Parameters

Find a particular solution to the inhomogeneous equation

$$y'' + y = \tan t$$

on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution: First, we find a fundamental set of solutions to $y'' + y = 0$. The characteristic polynomial is $f(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$, yielding $\lambda_1, \lambda_2 = \pm i$. Therefore, a fundamental set of solutions is $y_1(t) = \cos t$, $y_2(t) = \sin t$. Next, we compute the Wronskian:

$$W(t) = \cos^2 t + \sin^2 t = 1.$$

To find $v_1(t)$, we use:

$$\begin{aligned} v_1(t) &= \int \frac{-y_2(t)g(t)}{W(t)}\,dt \\ &= \int -\sin t \tan t\,dt \\ &= -\int \frac{\sin^2 t}{\cos t}\,dt \\ &= -\int \frac{1 - \cos^2 t}{\cos t}\,dt \end{aligned}$$

$$
\begin{aligned}
&= \sin t - \ln|\sec t + \tan t| \\
&= \sin t - \ln(\sec t + \tan t),
\end{aligned}
$$

since $\sec t + \tan t \geq 0$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. For $v_2(t)$, we find:

$$
\begin{aligned}
v_2(t) &= \int \frac{y_1(t)g(t)}{W(t)}\,dt \\
&= \int \cos t \tan t\,dt \\
&= \int \sin t \\
&= -\cos t.
\end{aligned}
$$

Thus, a particular solution is:

$$y_p(t) = y_1v_1 + y_2v_2 = \cos t(\sin t - \ln(\sec t + \tan t)) + \sin t(-\cos t) = -\cos t \ln(\sec t + \tan t).$$

8 Back to Linear Algebra

Matrix Vector Space Properties
Suppose $m, n \geq 1$, $A, B, C \in M_{m\times n}(\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}$. Then the following facts about matrix addition and scalar multiplication hold:

1. $(A + B) + C = A + (B + C)$ (associativity of addition)
2. $0_{m\times n} + A = A + 0_{m\times n} = A$ (additive identity)
3. $A + (-1)A = 0_{m\times n}$ (additive inverse)
4. $A + B = B + A$ (commutativity of addition)
5. $\alpha(A + B) = \alpha A + \alpha B$ (right distributivity)
6. $(\alpha + \beta)A = \alpha A + \beta A$ (left distributivity)
7. $(\alpha\beta)A = \alpha(\beta A)$ (associativity of scalar multiplication)
8. $1 \cdot A = A$ (here $1 \in \mathbb{R}$ is a scalar)

Homogeneous Matrix Equations
Suppose $A \in M_{m\times n}(\mathbb{R})$ and consider the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

where $\boldsymbol{b} \in \mathbb{R}^m$. We say that the equation is *homogeneous* if $\boldsymbol{b} = \boldsymbol{0}_{m\times 1}$ is the zero vector in $\mathbb{R}^m$. Otherwise, if $\boldsymbol{b} \neq \boldsymbol{0}$, then we say that the equation is *inhomogeneous*.

Nullspace
Suppose $A \in M_{m\times n}(\mathbb{R})$. We define the nullspace of A to be the following subset of $\mathbb{R}^n$:

$$\text{null}(A) := \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

In other words, the nullspace null(A) of the matrix A is the set of all solutions to the homogeneous equation $A\boldsymbol{x} = \boldsymbol{0}$. The nullspace is a subspace of the

Finding Nullspace Example

Find the nullspace of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Proof. We need to find the set of all vectors $\boldsymbol{x} \in \mathbb{R}^4$ such that $A\boldsymbol{x} = \mathbf{0}$. This means the same thing as finding all solutions to the system of equations:

$$\begin{aligned} x_1 + x_2 + 4x_4 &= 0 \\ x_3 + 2x_4 &= 0. \end{aligned}$$

To do this, we set up the system as an augmented matrix and take it to RREF:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

Here we see that the augmented matrix is already in RREF, so we can read off the solutions. We see that x_2, x_4 are free variables, so the general solution is:

$$\begin{aligned} x_1 &= -x_2 - 4x_4 \\ x_2 &= x_2 \\ x_3 &= -2x_4 \\ x_4 &= x_4 \end{aligned}$$

Which we can write in parametric form as a set of linear combinations of $\mathbb{R}^4$-vectors:

$$\text{null}(A) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

Basis of the Nullspace

Suppose $A \in M_{m\times n}(\mathbb{R})$. A basis of null($A$) is a collection of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$ such that:

1. null(A) = span($\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$) (so $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ can make all of null(A) by linear combinations), and
2. $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are linearly independent (so none of the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are unnecessary or redundant).

We define the dimension of null(A) to be the number of vectors in a basis of null(A). Thus

$$\dim \text{null}(A) := k \quad \Leftrightarrow \quad \text{there is a basis } \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \text{ of null}(A) \text{ with } k \text{ vectors.}$$

Notes Relating to the Rank-Nullity Theorem

Suppose $A \in M_{m\times n}(\mathbb{R})$:

1. In general, null(A) will have infinitely many possible bases, but all of these bases have the same size. Thus, the definition of dim null(A) does not depend on a particular choice of basis.
2. Recall that the rank of A (denoted rank(A)) is the number of pivots in the RREF of A. In general, dim null(A) is equal to the number of free variables in the RREF of A. Since the number of pivot variables plus the number of free variables, this yields the important rank-nullity formula:

$$\text{rank}(A) + \dim \text{null}(A) = n = \# \text{ of columns of } A.$$

3. A basis for null(A) can be obtained by solving the homogeneous equation $A\boldsymbol{c} = \mathbf{0}_{m\times 1}$ in the usual way with Gaussian Elimination, writing the solutions in parametric form with the free variables as parameters, then collecting each vector which gets multiplied by a free variable. This (finite) collection of vectors will be a basis for null(A).

Facts on Number of Solutions to a Linear System

Suppose $A \in M_{m\times n}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^m$.

1. The following are equivalent:
   (a) there does not exist any solutions to $A\boldsymbol{x} = \boldsymbol{b}$,
   (b) the system corresponding to $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent,
   (c) there does not exist a particular solution $\boldsymbol{x}_p$ to $A\boldsymbol{x} = \boldsymbol{b}$.

   We define the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ to be inconsistent if any of the equivalent conditions of (1) above. We say that $A\boldsymbol{x} = \boldsymbol{b}$ is consistent otherwise.

2. Suppose $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. The following are equivalent:
   (a) there is a unique solution to $A\boldsymbol{x} = \boldsymbol{b}$,
   (b) there is a unique solution to the system corresponding to $A\boldsymbol{x} = \boldsymbol{b}$,
   (c) $\text{null}(A) = \{\mathbf{0}_{n\times 1}\}$,
   (d) $\dim \text{null}(A) = 0$,
   (e) there are no free variables,
   (f) every variable is a pivot variable,
   (g) $\text{rank}(A) = n$.

3. Suppose $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. The following are equivalent:
   (a) there are infinitely many solutions to $A\boldsymbol{x} = \boldsymbol{b}$,
   (b) there are infinitely many solutions to the system corresponding to $A\boldsymbol{x} = \boldsymbol{b}$,
   (c) $\text{null}(A) \neq \{\mathbf{0}_{n\times 1}\}$,
   (d) $\dim \text{null}(A) \geq 1$,
   (e) there is at least one free variable,
   (f) $\text{rank}(A) < n$.

Determinant Properties

Suppose $A \in M_{n\times n}(\mathbb{R})$. Then the following are equivalent:

1. $\det(A) \neq 0$
2. $\text{null}(A) = \{\mathbf{0}\}$

Suppose $A, B \in M_{n\times n}(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then:

1. $\det(I_{n\times n}) = 1$
2. $\det(\alpha A) = \alpha^n \det(A)$
3. If B is obtained from A by either switching two rows or switching two columns (but not both), then $\det(B) = -\det(A)$.

Eigenvalues and Eigenvectors
Suppose $A \in M_{n\times n}(\mathbb{R})$ is a square matrix and $\lambda \in \mathbb{R}$. We say that λ is an eigenvalue for A if there exists a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$A\boldsymbol{v} = \lambda\boldsymbol{v}.$$

If λ is an eigenvalue of A, then we call a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ which satisfies

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

an eigenvector of A associated to λ.

Eigenvalue Theorem
Suppose $A \in M_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then the following are equivalent:

1. λ is an eigenvalue of A,
2. $\det(A - \lambda I) = 0$.

In other words, the eigenvalues of A are zeros of the "function" $\det(A - \lambda I)$. As it turns out, the expression $\det(A - \lambda I)$ is always a polynomial in the variable λ. This polynomial has a special name, the characteristic polynomial.

This result follows from the equivalences below:

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } A &\Leftrightarrow \text{there exists nonzero } \boldsymbol{v} \in \mathbb{R}^n \text{ such that } A\boldsymbol{v} = \lambda\boldsymbol{v} \\
&\Leftrightarrow \text{there exists nonzero } \boldsymbol{v} \in \mathbb{R}^n \text{ such that } A\boldsymbol{v} - \lambda\boldsymbol{v} = \mathbf{0} \\
&\Leftrightarrow \text{there exists nonzero } \boldsymbol{v} \in \mathbb{R}^n \text{ such that } A\boldsymbol{v} - \lambda I\boldsymbol{v} = \mathbf{0} \\
&\Leftrightarrow \text{there exists nonzero } \boldsymbol{v} \in \mathbb{R}^n \text{ such that } (A - \lambda I)\boldsymbol{v} = \mathbf{0} \\
&\Leftrightarrow \text{null}(A - \lambda I) \neq \{\mathbf{0}\} \\
&\Leftrightarrow \det(A - \lambda I) = 0, \text{ by the Determinant Property above.}
\end{aligned}$$

Characteristic Polynomial
Suppose $A \in M_{n\times n}(\mathbb{R})$. The polynomial

$$p(\lambda) := (-1)^n \det(A - \lambda I) = \det(\lambda I - A)$$

is called the characteristic polynomial of A, and the equation

$$p(\lambda) = 0$$

is called the characteristic equation.
Note: the factor $(-1)^n$ ensures that the polynomial is monic, i.e. has a positive leading coefficient $\implies 1$

Thus, the Eigenvalue Theorem states that the eigenvalues of A are precisely the zeros of its characteristic polynomial.

Eigenspace
Suppose $A \in M_{n\times n}(\mathbb{R})$ and λ is an eigenvalue of A. We define the eigenspace of λ to be

$$E_\lambda := \text{null}(A - \lambda I),$$

i.e., the eigenspace E_λ is the set of all eigenvectors associated to λ together with the zero vector.

Example Finding the Eigenspace
Find all eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. We found three distinct eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 4$. For each of these eigenvalues, we need to compute a basis of its eigenspace.

$\boldsymbol{\lambda_1 = 1}$: We will compute a basis of

$$\text{null}(A - I) = \text{null}\left(\begin{bmatrix} 3 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right).$$

Note that

$$\begin{bmatrix} 3 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that x_2 is a free variable and thus the general solution is:

$$x_1 = 0, \quad x_2 = x_2, \quad x_3 = 0.$$

Thus we can express the eigenspace E_1 as

$$E_1 = \text{null}(A - I) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

$\boldsymbol{\lambda_2 = 2}$: We compute a basis of null$(A - 2I)$:

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that x_3 is a free variable and the general solution is

$$x_1 = x_3, \quad x_2 = 3x_3, \quad x_3 = x_3.$$

Thus we can express the eigenspace E_2 as

$$E_2 = \text{null}(A - 2I) = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\}.$$

$\boldsymbol{\lambda_3 = 4}$: We compute a basis of null$(A - 4I)$:

$$\begin{bmatrix} 0 & 0 & -2 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that x_2 is a free variable and the general solution is

$$x_1 = 3x_2, \quad x_2 = x_2, \quad x_3 = 0.$$

Thus we can express the eigenspace E_4 as

$$E_4 = \text{null}(A - 4I) = \text{span}\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Facts on Eigenbases

Suppose $A \in M_{n\times n}(\mathbb{R})$ is a square matrix. An eigenbasis of A is a basis of $\mathbb{R}^n$ which is composed of eigenvectors of A. In other words, a set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$ is an eigenbasis of A if

1. $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ for some λ_i, for each $i = 1, \dots, n$.
2. $\mathbb{R}^n = \text{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$
3. $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent.

Here is a fact about eigenbases which we are happy to assume: Suppose $A \in M_{n\times n}(\mathbb{R})$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$, for some $k \leq n$. If

1. β_i is a basis of E_{λ_i} for each $i = 1, \dots, k$ and
2. $|\beta_1| + |\beta_2| + \cdots + |\beta_k| = n$,

then $\beta := \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an eigenbasis of A. In particular, if $k = n$, then $\beta = \beta_1 \cup \cdots \cup \beta_n$ is always an eigenbasis (i.e., condition (2) is automatically satisfied).

9 Linear Systems

Homogeneous Linear System of Differential Equations

A homogeneous linear system of differential equations (with constant coefficients) is a set of differential equations of the following form:

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t)\\
x_2'(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t)\\
&\vdots\\
x_n'(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t)
\end{aligned}$$

where each $a_{ij} \in \mathbb{R}$ and $x_1(t), \dots, x_n(t)$ are unknown functions. A solution to the system is a collection of n differentiable functions $x_1, x_2, \dots, x_n : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) such that plugging these functions into the system makes each equation true.

We will prefer to write systems in terms of matrices and vectors, so we can rewrite the system above as:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

or even as:

$$\boldsymbol{x}' = A\boldsymbol{x}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and

$$\boldsymbol{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Note that with this notation, a solution is now a vector-valued function $\boldsymbol{x}(t) : \mathbb{R} \to \mathbb{R}^n$.

Linear Independence of Vector-Valued Functions

Suppose $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_k(t) : I \to \mathbb{R}^n$ are vector-valued functions. We say that $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_k(t)$ are linearly independent if for every $c_1, \ldots, c_k \in \mathbb{R}$, if

$$c_1\boldsymbol{x}_1(t) + \cdots + c_k\boldsymbol{x}_k(t) = \boldsymbol{0} \text{ for all } t \in I,$$

then $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, we say that $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_k(t)$ are linearly dependent.

Fact: Suppose $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_k(t)$ are solutions to $\boldsymbol{x}' = A\boldsymbol{x}$. If there is some fixed t_0 such that the column vectors $\boldsymbol{x}_1(t_0), \ldots, \boldsymbol{x}_k(t_0) \in \mathbb{R}^n$ are linearly dependent (respectively, linearly independent), then the functions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_k(t)$ are linearly dependent (resp., linearly independent). This means just check a point.

Fundamental Set of Solutions for Linear System

Suppose $A \in M_{n\times n}(\mathbb{R})$ and $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ are n linearly independent solutions to

$$\boldsymbol{x}' = A\boldsymbol{x}.$$

Then $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ form a fundamental set of solutions, i.e., if $\boldsymbol{x}_0(t)$ is an arbitrary solution, then there are (necessarily unique) $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\boldsymbol{x}_0(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \cdots + c_n\boldsymbol{x}_n(t) \text{ for every } t.$$

Finding a Solution to Linear System

Suppose $A \in M_{n\times n}(\mathbb{R})$, λ is an eigenvalue of A, and $\boldsymbol{v}$ is an eigenvector associated to λ. Then

$$\boldsymbol{x}(t) := e^{\lambda t}\boldsymbol{v}$$

is a solution to the system $\boldsymbol{x}' = A\boldsymbol{x}$ and satisfies the initial condition $\boldsymbol{x}(0) = \boldsymbol{v}$.

Proof: Let $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}$ be as in the statement of the proposition. Note that the lefthand side yields:

$$\boldsymbol{x}'(t) = (e^{\lambda t}\boldsymbol{v})' = (\lambda e^{\lambda t})\boldsymbol{v} = \lambda e^{\lambda t}\boldsymbol{v} = \lambda\boldsymbol{x}(t).$$

Whereas the righthand side yields:

$$A\boldsymbol{x}(t) = A(e^{\lambda t}\boldsymbol{v}) = e^{\lambda t}A\boldsymbol{v} = e^{\lambda t}\lambda\boldsymbol{v} = \lambda\boldsymbol{x}(t).$$

Planar System

We will consider the case where $A \in M_{2\times 2}(\mathbb{R})$

Planar System with Distinct Real Roots

Suppose $A \in M_{2\times 2}(\mathbb{R})$ has two distinct real eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$. Furthermore, suppose $\boldsymbol{v}_1$ is an eigenvector associated with λ_1 and $\boldsymbol{v}_2$ is an eigenvector associated with λ_2. Then the general solution to

$$\boldsymbol{x}' = A\boldsymbol{x}$$

is

$$\boldsymbol{x}(t; C_1, C_2) = C_1e^{\lambda_1 t}\boldsymbol{v}_1 + C_2e^{\lambda_2 t}\boldsymbol{v}_2.$$

Complex Roots Case

Suppose $A \in M_{2\times 2}(\mathbb{R})$ has complex conjugate eigenvalues $\lambda, \overline{\lambda} \notin \mathbb{R}$, and $\boldsymbol{w}$ is an eigenvector associated to λ. Then $\boldsymbol{w}$ is also associated with $\overline{\lambda}$. Furthermore:

1) (Complex version) The general solution to $\boldsymbol{x}' = A\boldsymbol{x}$ in terms of complex-valued functions is:

$$\boldsymbol{x}(t; C_1, C_2) = C_1e^{\lambda t}\boldsymbol{w} + C_2e^{\overline{\lambda} t}\overline{\boldsymbol{w}}$$

2) (Real version) The general solution to $\boldsymbol{x}' = A\boldsymbol{x}$ in terms of real-valued functions is:

$$\boldsymbol{x}(t; C_1, C_2) = C_1 e^{\alpha t}(\cos(\beta t)\boldsymbol{v}_1 - \sin(\beta t)\boldsymbol{v}_2) + C_2 e^{\alpha t}(\sin(\beta t)\boldsymbol{v}_1 + \cos(\beta t)\boldsymbol{v}_2)$$

where $\lambda = \alpha + i\beta$ and $\boldsymbol{w} = \boldsymbol{v}_1 + i\boldsymbol{v}_2$.

Double Real Root Case Easy Version

Suppose $A \in M_{2\times 2}(\mathbb{R})$ has only one eigenvalue $\lambda \in \mathbb{R}$ (of multiplicity two). Furthermore, suppose we can find two linearly independent eigenvectors of A associated to λ. Then the general solution to $\boldsymbol{x}' = A\boldsymbol{x}$ is

$$\boldsymbol{x}(t; C_1, C_2) = C_1 e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix} + C_2 e^{\lambda t}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}C_1 e^{\lambda t}\\C_2 e^{\lambda t}\end{bmatrix}.$$

Proof. Let

$$\boldsymbol{x}_1(t) := e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix}, \quad \boldsymbol{x}_2(t) := e^{\lambda t}\begin{bmatrix}0\\1\end{bmatrix}.$$

By assumption, the eigenspace of λ is two-dimensional, so it must be all of $\mathbb{R}^2$, Thus the following two vectors are eigenvectors associated to λ:

$$\boldsymbol{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

Both $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ are solutions to $\boldsymbol{x}' = A\boldsymbol{x}$. Furthermore, since $\boldsymbol{x}_1(0)$, $\boldsymbol{x}_2(0)$ are linearly independent, it follows that $\boldsymbol{x}_1(t)$, $\boldsymbol{x}_2(t)$ are also linearly independent. Thus by our earlier theorem, it follows that the general solution is

$$\boldsymbol{x}(t; C_1, C_2) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) = C_1 e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix} + C_2 e^{\lambda t}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}C_1 e^{\lambda t}\\C_2 e^{\lambda t}\end{bmatrix}.$$

Note: this occurs when our matrix is in the following form where $\lambda \in \mathbb{R}$:

$$\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}$$

It can be thought of as two distinct first order linear homogeneous differential equations with simple solutions that we have worked with earlier.

Double Real Root Interesting Case

Suppose $A \in M_{2\times 2}(\mathbb{R})$ has only one eigenvalue $\lambda \in \mathbb{R}$ (of multiplicity two). Furthermore, suppose we can only find one linearly independent eigenvector $\boldsymbol{v}_1$ of A associated to λ. Then the general solution to $\boldsymbol{x}' = A\boldsymbol{x}$ is

$$\boldsymbol{x}(t; C_1, C_2) = C_1 e^{\lambda t}\boldsymbol{v}_1 + C_2 e^{\lambda t}(\boldsymbol{v}_2 + t\boldsymbol{v}_1)$$

where $\boldsymbol{v}_2 \in \mathbb{R}^2$ is any particular solution to the matrix equation $(A - \lambda I)\boldsymbol{v}_2 = \boldsymbol{v}_1$.

Reasoning. Given the single linearly independent eigenvector $\boldsymbol{v}_1$, we look for a general solution that incorporates the eigenvector and accounts for the lack of a second independent eigenvector by introducing $\boldsymbol{v}_2$, which satisfies the matrix equation $(A-\lambda I)\boldsymbol{v}_2 = \boldsymbol{v}_1$. The term $C_2 e^{\lambda t} t\boldsymbol{v}_1$ compensates for the dimensional shortfall of the eigenspace associated with λ, ensuring the solution space is fully spanned. This construction leverages the Jordan chain concept, where $\boldsymbol{v}_2$ can be considered a generalized eigenvector. The resulting expression for $\boldsymbol{x}(t; C_1, C_2)$ thereby satisfies the differential system $\boldsymbol{x}' = A\boldsymbol{x}$ for any initial conditions encapsulated by C_1 and C_2.

Double Real Root Interesting Case Example

Find the general solution to $\boldsymbol{x}' = A\boldsymbol{x}$, where

$$A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2.$$

Thus, $\lambda_1 = 1$ is the only eigenvalue (of multiplicity two). For null$(A - I)$, we find that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is already in RREF, indicating only one linearly independent eigenvector:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Given $\boldsymbol{v}_1$, a solution to $\boldsymbol{x}' = A\boldsymbol{x}$ is

$$\boldsymbol{x}_1(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For the second solution, we consider the form

$$\boldsymbol{x}(t) = e^{\lambda_1 t}(\boldsymbol{v}_2 + t\boldsymbol{v}_1)$$

and find $\boldsymbol{v}_2$ satisfying

$$(A - \lambda_1 I)\boldsymbol{v}_2 = \boldsymbol{v}_1.$$

Solving gives $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, leading to

$$\boldsymbol{x}_2(t) = e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = e^t \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

The general solution is

$$\boldsymbol{x}(t; C_1, C_2) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^t \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 e^t + C_2 t e^t \\ C_2 e^t \end{bmatrix}.$$

9.1 Higher Order Differential Equations

Homogeneous nth Order Linear Differential Equations

A homogeneous nth order linear differential equation with constant coefficients is given by:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$$

where $a_1, \ldots, a_n \in \mathbb{R}$.

The solution to these equations involves a three-step process:

1. Convert the nth order linear system (in one unknown function) to an $n \times n$ linear system (with n unknown functions).
2. Solve the nth order linear system.
3. Convert the solution back in terms of a solution of the original linear differential equation.

Theorems on Homogeneous Higher Order Linear Systems
Consider the nth order homogeneous linear differential equation with constant coefficients:

$$(A) \quad y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \quad \text{with } a_1, \ldots, a_n \in \mathbb{R}.$$

and let

$$(B) \quad \boldsymbol{x}' = A\boldsymbol{x}$$

be the associated linear system.

(1) The following are equivalent:

(a) $y(t)$ is a solution to (A)

(b) the vector-valued function $\boldsymbol{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$ is a solution to (B).

(2) Suppose $y_1(t), \ldots, y_n(t)$ are solutions to (A). The following are equivalent:

(a) $y_1(t), \ldots, y_n(t)$ are linearly independent (as real-valued functions)

(b) The vector-valued functions $\begin{bmatrix} y_1(t) \\ y_1'(t) \\ \vdots \\ y_1^{(n-1)}(t) \end{bmatrix}, \ldots, \begin{bmatrix} y_n(t) \\ y_n'(t) \\ \vdots \\ y_n^{(n-1)}(t) \end{bmatrix}$ are linearly independent.

(c) For some t_0 the determinant $\det \begin{bmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{bmatrix} \neq 0.$

(d) For every t, the determinant $\det \begin{bmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix} \neq 0.$

Wronskian for Higher Order Differential Equations
Let $y_1, \ldots, y_n : I \to \mathbb{R}$ be real-valued functions (where $I \subseteq \mathbb{R}$ is an interval). The Wronskian of $y_1, \ldots, y_n$ is defined to be the function

$$W(t) = \det \begin{bmatrix} y_1(t) & \cdots & y_n(t) \\ y_1'(t) & \cdots & y_n'(t) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}.$$

Example Solving Higher Order Linear Differential Equation
Find the general solution to:

$$y^{(4)} - 13y'' + 36y = 0.$$

Solution. This equation involves one unknown function. First, we convert this into an equation with four unknown functions by introducing three auxiliary variables. Defining $x_2(t) := y'(t)$, $x_3(t) := y''(t) = x_2'(t)$, and $x_4(t) := y'''(t) = x_3'(t)$, and for uniform notation, $x_1(t) := y(t)$, we get:

$$x_1'(t) = x_2(t),$$

$$x_2'(t) = x_3(t),$$
$$x_3'(t) = x_4(t),$$
$$x_4'(t) = 13x_3(t) - 36x_1(t).$$

This leads to the system:

$$\boldsymbol{x}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & 13 & 0 \end{bmatrix} \boldsymbol{x}(t).$$

To solve, we first compute the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = \lambda^4 - 13\lambda^2 + 36 = (\lambda - 2)(\lambda + 2)(\lambda - 3)(\lambda + 3).$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 3$, $\lambda_4 = -3$. The corresponding eigenvectors (calculation omitted) are:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -4 \\ 8 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} -1 \\ 3 \\ -9 \\ 27 \end{bmatrix}.$$

Thus, the general solution to $\boldsymbol{x}' = A\boldsymbol{x}$ is:

$$\boldsymbol{x}(t; C_1, C_2, C_3, C_4) = C_1 e^{2t}\boldsymbol{v}_1 + C_2 e^{-2t}\boldsymbol{v}_2 + C_3 e^{3t}\boldsymbol{v}_3 + C_4 e^{-3t}\boldsymbol{v}_4.$$

In particular, the general solution to $y^{(4)} - 13y'' + 36y = 0$ is:

$$y(t) = C_1 e^{2t} + C_2 e^{-2t} + C_3 e^{3t} + C_4 e^{-3t}.$$

General Strategy & The Companion Matrix

In general, for an nth order homogeneous linear differential equation with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \quad \text{with } a_1, \ldots, a_n \in \mathbb{R},$$

we introduce $n - 1$ additional unknown functions for the higher derivatives of y: $x_1(t) := y(t)$, $x_2(t) := x_1'(t) = y'(t)$, $x_3(t) := x_2'(t) = y''(t)$, $\ldots$, $x_n(t) := x_{n-1}'(t) = y^{(n-1)}(t)$. This leads to the system:

$$x_1'(t) = x_2(t),$$
$$x_2'(t) = x_3(t),$$
$$\vdots$$
$$x_{n-1}'(t) = x_n(t).$$

The relation $x_n'(t) = y^{(n)}(t)$ can be connected to the lower derivatives using the original differential equation:

$$x_n'(t) = -a_n x_1(t) - a_{n-1} x_2(t) - \cdots - a_1 x_n(t).$$

Formulating the linear system:

$$\boldsymbol{x}'(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}(t).$$

Here, the matrix A is known as the ***companion matrix*** of the linear differential equation, giving us an $n \times n$ linear system of the form $\boldsymbol{x}' = A\boldsymbol{x}$.