32BH Notes

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Upper and Lower Darboux Sums → Integral

Bounded Subset
A subset $D \subset \mathbb{R}^n$ is bounded if there exists some $r > 0$ such that $D \subset B_r(\mathbf{0})$.

Bounded Function
A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is bounded if its image $\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in A\}$ is a bounded subset of $\mathbb{R}$.

Boundary Point
A point $\boldsymbol{x} \in \mathbb{R}^n$ is a boundary point of $D \subset \mathbb{R}^n$ if: for all $\varepsilon > 0$,

1. $B_\varepsilon(\boldsymbol{x}) \cap D$ is non-empty, and
2. $B_\varepsilon(\boldsymbol{x}) \cap D^c$ is non-empty.

Closure
The closure of a set $D \subset \mathbb{R}^n$ is the union of D and the boundary of D. That is, the closure is the set

$$\overline{D} = \{\boldsymbol{x} \in \mathbb{R}^n \mid B_r(\boldsymbol{x}) \cap D \neq \emptyset \text{ for all } r > 0\}$$

Support of a Function
The support of a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is the closure of the set of non-zero values of a function:

$$\operatorname{supp}(f) := \overline{\{\boldsymbol{x} \in A \mid f(\boldsymbol{x}) \neq 0\}}$$

Bounded Support
A function has bounded support if its support is bounded. Equivalently, there exists $R > 0$ such that $f(\boldsymbol{x}) = 0$ for all $\|\boldsymbol{x}\| > R$.

Partition
A partition of a set X is a collection of non-empty subsets $X_\alpha \subset X$ such that every element of $x \in X$ is in exactly one X_α.

Dyadic Cubes
Given a vector $\boldsymbol{k} = \langle k_1, \ldots, k_n \rangle \in \mathbb{Z}^n \subset \mathbb{R}^n$ (that is, $k_i \in \mathbb{Z}$ for all i), we can define the dyadic cube $C_{\boldsymbol{k},N}$ in $\mathbb{R}^n$ as

$$C_{\boldsymbol{k},N} := \left\{ \boldsymbol{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n \mid \frac{k_i}{2^N} \leq x_i < \frac{k_i + 1}{2^N} \text{ for all } i \right\}$$

$$\text{For a fixed } N, \text{ the collection of all dyadic cubes } D_N(\mathbb{R}^n) := \{C_{\boldsymbol{k},N} \mid \text{for all } \boldsymbol{k} \in \mathbb{Z}^n\}$$

$$\text{The volume of a dyadic cube } C_{\boldsymbol{k},N} \text{ in } \mathbb{R}^n \text{ is } \frac{1}{2^{Nn}}$$

Upper Bound

Let $X \subset \mathbb{R}$. A number $M \in \mathbb{R}$ is an upper bound of X if for every $x \in X$, we have that $x \leq M$.

Lower Bound

Let $X \subset \mathbb{R}$. A number $m \in \mathbb{R}$ is a lower bound of X if for every $x \in X$, we have that $m \leq x$.

Supremum

Let q be an upper bound of X. We say q is the supremum of X (or least upper bound of X) if for all upper bounds M of X, we have that $q \leq M$. We write $q := \sup(X)$. If X is not bounded above, we write $\sup(X) = \infty$.

Infimum

Let p be a lower bound of X. We say p is the infimum of X (or greatest lower bound of X) if for all lower bounds m of X, we have that $m \leq p$. We write $p := \inf(X)$. If X is not bounded below, we write $\inf(X) = -\infty$.

Supremum and Infimum of a Function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and $D \subset \mathbb{R}^n$ an arbitrary subset. We will consider the following quantities:

$$M_D(f) := \sup\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in D\}$$

$$m_D(f) := \inf\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in D\}$$

Darboux Sums

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. The N-th upper Darboux sum and N-th lower Darboux sum of f are defined as follows:

$$U_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} M_C(f) \cdot \text{vol}(C) = \frac{1}{2^{Nn}} \sum_{C \in D_N(\mathbb{R}^n)} M_C(f)$$

$$L_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} m_C(f) \cdot \text{vol}(C) = \frac{1}{2^{Nn}} \sum_{C \in D_N(\mathbb{R}^n)} m_C(f)$$

Darboux Integrals

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. The upper Darboux integral and lower Darboux integral of f are defined as

$$U(f) := \lim_{N \to \infty} U_N(f)$$

$$L(f) := \lim_{N \to \infty} L_N(f)$$

Definition of Integrability

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. We say that f is integrable if $U(f) = L(f)$. The integral of f is defined as

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\, dV := U(f) = L(f)$$

This is equivalent to stating that for any $\varepsilon > 0$, there exists N such that

$$|U_N(f) - L_N(f)| < \varepsilon$$

Indicator Function, Extensions, and Integrals

Let $B \subset \mathbb{R}^n$ be a subset. The indicator function $1_B : \mathbb{R}^n \to \mathbb{R}$ is the function defined by

$$1_B(\boldsymbol{x}) := \begin{cases} 1 & \text{if } \boldsymbol{x} \in B \\ 0 & \text{if } \boldsymbol{x} \notin B \end{cases}$$

And given a function $f : \mathbb{R}^n \to \mathbb{R}$, then the function $f(\boldsymbol{x})1_B(\boldsymbol{x})$ is the piecewise function defined by

$$f(\boldsymbol{x})1_B(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in B \\ 0 & \text{if } \boldsymbol{x} \notin B \end{cases}$$

Furthermore, let $A \subset \mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a function. We can extend f to a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by defining

$$\tilde{f}(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in A \\ 0 & \text{if } \boldsymbol{x} \notin A \end{cases}$$

We will often use the following abusive notation when we want to indicate the domain A:

$$\tilde{f}(\boldsymbol{x})1_A(\boldsymbol{x}) := f(\boldsymbol{x})$$

Taken together, we have the following:

Let $B \subset \mathbb{R}^n$, and let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be an integrable function. Then we can define the integral of f over B as

$$\int_B f(\boldsymbol{x})\, dV := \int_{\mathbb{R}^n} \tilde{f}(\boldsymbol{x})1_A(\boldsymbol{x})1_B(\boldsymbol{x})\, dV$$

By construction, we have the properties of the integral:

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\, dV = \int_B \tilde{f}(\boldsymbol{x})\, dV = \int_{\mathbb{R}^n} \tilde{f}(\boldsymbol{x})1_B(\boldsymbol{x})\, dV = \int_{\mathbb{R}^n} \tilde{f}(\boldsymbol{x})1_A(\boldsymbol{x})1_B(\boldsymbol{x})\, dV$$

Properties of Integrals

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two integrable functions. Then

(a) $f + g$ is also integrable, and

$$\int_{\mathbb{R}^n} (f+g)\, dV = \int_{\mathbb{R}^n} f\, dV + \int_{\mathbb{R}^n} g\, dV$$

(b) If $\lambda \in \mathbb{R}$, then λf is integrable, and

$$\int_{\mathbb{R}^n} \lambda f\, dV = \lambda \int_{\mathbb{R}^n} f\, dV$$

(c) If $f(\boldsymbol{x}) \leq g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f\, dV \leq \int_{\mathbb{R}^n} g\, dV$$

(d) $|f|(\boldsymbol{x}) := |f(\boldsymbol{x})|$ is integrable, and

$$\left| \int_{\mathbb{R}^n} f\, dV \right| = \int_{\mathbb{R}^n} |f|\, dV$$

Auxiliary Functions f^+ and f^-
Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we define two auxiliary non-negative functions, f^+ and f^-.

$$f^+(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}) & \text{if } f(\boldsymbol{x}) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^-(\boldsymbol{x}) := \begin{cases} -f(\boldsymbol{x}) & \text{if } f(\boldsymbol{x}) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Calculating Multivariable Integrals

Fubini's Theorem
Let $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function that is bounded with bounded support, and let $(i_1, \ldots, i_n)$ be a permutation of the set $\{1, \ldots, n\}$. Then

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\, dV = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\boldsymbol{x})\, dx_{i_1} \cdots dx_{i_n}$$

That is, we can compute an integral over $\mathbb{R}^n$ of $f(\boldsymbol{x})\, dV$ as an iterated integral, in any variable order!

Decomposition of Domains
Let K be a compact (closed and bounded) subset in $\mathbb{R}^n$ such that its boundary ∂K has volume zero. Furthermore, let $K = K_1 \cup K_2$, such that K_1 and K_2 are compact, and the intersection $K_1 \cap K_2$ has volume zero.

Let $f : K \to \mathbb{R}$ be a continuous function. Then f is integrable over K_1 and K_2, and

$$\int_K f(\boldsymbol{x})\, dA = \int_{K_1} f(\boldsymbol{x})\, dA + \int_{K_2} f(\boldsymbol{x})\, dA$$

Volume of Integrating Region
Let $A \subset \mathbb{R}^n$. If $1_A : \mathbb{R}^n \to \mathbb{R}$ is integrable, then the n-dimensional volume of A is given by

$$\text{vol}_n(A) := \int_{\mathbb{R}^n} 1_A\, dV$$

$n + 1$ Dimensional Volume of the Graph Γ_f
If $X \subset \mathbb{R}^n$ is a closed and bounded (**compact**) region and $f : X \to \mathbb{R}$ is a continuous function, then the $(n + 1)$-dimensional volume of the graph Γ_f is 0.

Product of Integrals
Suppose that $f(\boldsymbol{x})$ is integrable on $\mathbb{R}^n$, and $g(\boldsymbol{y})$ is integrable on $\mathbb{R}^m$. Then $h(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x})g(\boldsymbol{y})$ is integrable on $\mathbb{R}^{n+m}$, and

$$\int_{\mathbb{R}^{n+m}} h\, dV\, dW = \left(\int_{\mathbb{R}^n} f\, dV \right) \left(\int_{\mathbb{R}^m} g\, dW \right)$$

Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ must be different variables

Vertically Simple
A subset $D \subset \mathbb{R}^2$ is vertically simple if it is the region between the graphs of two continuous functions $y = g_1(x)$ and $y = g_2(x)$ over a fixed interval of x-values $[a, b]$.

Horizontally Simple
A subset $D \subset \mathbb{R}^2$ is horizontally simple if it is the region between the graphs of two continuous functions $x = h_1(y)$ and $x = h_2(y)$ over a fixed interval of y-values $[c, d]$.

Oscillation of a Function
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and let $A \subset \mathbb{R}^n$. The oscillation of f over A is defined as

$$\mathrm{osc}_A(f) := M_A(f) - m_A(f)$$

Open Ball
An open ball $B \subset \mathbb{R}^n$ of radius $\delta > 0$, centered on $\boldsymbol{x}$, is the set

$$B = \{\boldsymbol{v} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{v}\| < \delta\}$$

Measure of a Set
A set $X \subset \mathbb{R}^n$ has measure zero if for every $\varepsilon > 0$, there exists an infinite sequence of open balls B_i such that

$$X \subset \bigcup_i B_i \text{ and } \sum_i \mathrm{vol}_n(B_i) < \varepsilon$$

Note: A set of volume 0 has measure zero, but on the other hand, it is possible that X has measure zero, but $\mathrm{vol}(X)$ is undefined.

Expansion of Definition of Integrability
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. Then the following are equivalent:

(a) f is integrable.

(b) For any $\varepsilon > 0$, there exists N such that for all $n > N$, $U_n(f) - L_n(f) < \varepsilon$.

(c) For any $\varepsilon > 0$, there exists N such that, $U_N(f) - L_N(f) < \varepsilon$.

$$\Downarrow$$

Integrability Criterion I: A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if

(a) f is bounded with bounded support,

(b) For all $\varepsilon > 0$, there exists N such that

$$\sum_{\{C \in D_N(\mathbb{R}^n) \mid \mathrm{osc}_C(f) > \varepsilon\}} \mathrm{vol}_n(C) < \varepsilon$$

$$\Downarrow$$

Integrability Criterion II:Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. If f is continuous except on a set of volume zero, then f is integrable.

$$\Downarrow$$

Integrability Criterion III: A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if

(a) f is bounded with bounded support

(b) f is continuous except on a set of measure 0

Volume Zero
A bounded set $X \subset \mathbb{R}^n$ has n-dimensional volume 0 if and only if for every $\epsilon > 0$, there exists M such that

$$\sum_{C \in D_M(\mathbb{R}^n) | C \cap X \neq \emptyset} \text{vol}_n(C) < \varepsilon$$

If $X \subset \mathbb{R}^n$ is a closed and bounded (compact) region, and $f : X \to \mathbb{R}$ is a continuous function, then

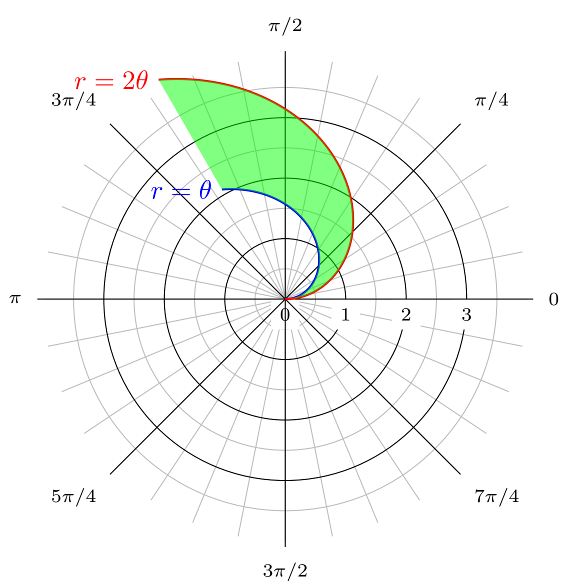
$$\text{vol}_{n+1}(\Gamma_f) = 0$$

Polar, Cylindrical, and Spherical Coordinates:

Radially Simple
A region R is called radially simple if it is the region between the graphs of two continuous functions $r_1(\theta)$ and $r_2(\theta)$ over a fixed interval of θ-values. That is,

$$R = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\, r_1(\theta) \leq r \leq r_2(\theta)\}$$

Consider:



Double Integral in Polar Coordinates
If $f(x, y)$ is a continuous function on a radially simple domain R, then the double integral of f over R in polar coordinates is given by

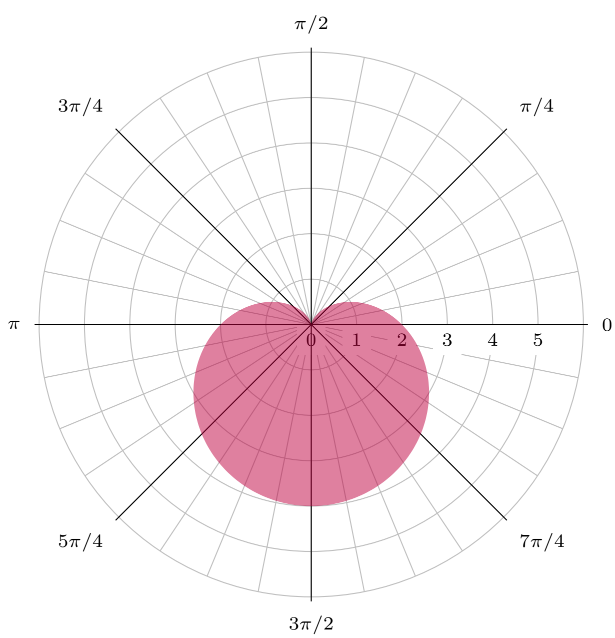
$$\int\int_R f(x, y)\, dA = \int_\alpha^\beta \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Note the additional r term which is the result of the general change of variables formula.

Consider an example:

$$r = 2 - 2\sin(\theta)$$

$$\begin{aligned}
A &= \int_0^{2\pi} \int_0^{2-2\sin(\theta)} r\,dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_0^{2-2\sin(\theta)} d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(2-2\sin(\theta))^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (4 - 8\sin(\theta) + 4\sin^2(\theta))\,d\theta \\
&= \frac{1}{2}\left(\int_0^{2\pi} 4\,d\theta - \int_0^{2\pi} 8\sin(\theta)\,d\theta + \int_0^{2\pi} 4\sin^2(\theta)\,d\theta\right) \\
&= \frac{1}{2}\left(4\theta\Big|_0^{2\pi} - 8(-\cos(\theta))\Big|_0^{2\pi} + 4\int_0^{2\pi} \sin^2(\theta)\,d\theta\right) \\
&= \frac{1}{2}\left(8\pi + 4\int_0^{2\pi} \sin^2(\theta)\,d\theta\right) \\
&= \frac{1}{2}\left(8\pi + 4\int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\,d\theta\right) \quad \text{(using the identity } \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}\text{)} \\
&= \frac{1}{2}\left(8\pi + 4\pi - \int_0^{2\pi} \cos(2\theta)\,d\theta\right) \\
&= \frac{1}{2}(12\pi - 0) \quad \text{(since the integral of } \cos(2\theta) \text{ over a full period is 0)} \\
&= 6\pi
\end{aligned}$$



Rectangular to Cylindrical Coordinates

Given a point (x, y, z) in Euclidean coordinates, we can convert it to a point (r, θ, z) in cylindrical coordinates by setting

$$z = z, \quad r = \sqrt{x^2 + y^2}, \quad \tan(\theta) = \frac{y}{x}$$

(assuming $x \neq 0$).

Cylindrical to Rectangular Coordinates
Given a point (r, θ, z) in cylindrical coordinates, we can convert it to a point (x, y, z) in rectangular coordinates by setting

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Rectangular to Spherical Coordinates
Given a point (x, y, z) in standard Euclidean coordinates, we can convert it to spherical coordinates by setting

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan(\theta) = \frac{y}{x}, \quad \cos(\phi) = \frac{z}{\rho}$$

Spherical to Rectangular Coordinates
Given a point (ρ, θ, ϕ) in spherical coordinates, we can convert it to standard Euclidean coordinates by setting

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)$$

Centrally Simple
A solid region $R \subset \mathbb{R}^3$ is called centrally simple if R is of the form

$$R = \{(\rho, \theta, \phi) \mid \theta_1 \leq \theta \leq \theta_2,\ \phi_1 \leq \phi \leq \phi_2,\ \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)\}$$

Triple Integrals in Spherical Coordinates
Let $f(x, y, z)$ be a continuous function on a centrally simple region R. Define

$$g(\theta, \phi, \rho) = f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$

Then the integral of f over R is given by

$$\int\int\int_R f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho=\rho_1(\theta,\phi)}^{\rho=\rho_2(\theta,\phi)} g(\theta, \phi, \rho)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Like with double integrals in polar, there is an extra term $\rho^2 \sin(\phi)$ from the general change of variables formula.

Linear Algebra Review

Linear Maps
A linear map $T : V \to W$ is defined as follows for all $k \in \mathbb{N}, \alpha_i \in \mathbb{R}$, and all vectors $\boldsymbol{x}_i \in V$:

$$T\left(\sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i\right) = \sum_{i=1}^{k} \alpha_i T(\boldsymbol{x}_i)$$

Equivalently, a linear map will satisfy the following:

i $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$

ii $T(\lambda \boldsymbol{u}) = \lambda T(\boldsymbol{u})$

Additionally, a map $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there is a matrix A in $M_{m\times n}(\mathbb{R})$ such that

$$T(\boldsymbol{x}) = A\boldsymbol{x}$$

We call A the standard matrix of T.

Standard Matrix
Given a basis $\mathcal{B} = \{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$, the standard matrix A of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$\big[A\big] = \begin{bmatrix} | & | & & | \\ T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \cdots & T(\boldsymbol{e_n}) \\ | & | & & | \end{bmatrix} \in M_{m\times n}(\mathbb{R})$$

Area of a Parallelogram
Let P be the parallelogram spanned by $\boldsymbol{u} = \langle A, B\rangle$ and $\boldsymbol{v} = \langle C, D\rangle$ in $\mathbb{R}^2$.

$$\text{area}(P) = \left|\det \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|$$

That is, the absolute value of a 2×2 determinant equals the area of the parallelogram spanned by the rows.

Volume of a Parallelepiped
Let D be the parallelepiped spanned by vectors $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ in $\mathbb{R}^3$. Then the volume of D is given by the absolute value of the scalar triple product:

$$\text{volume}(D) = |\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})| = \left|\det \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}\right|$$

This generalizes in the following way. Let D be the parallelepiped spanned by vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$ in $\mathbb{R}^n$. Then the n-dimensional volume of D is given by the absolute value of the determinant:

$$\text{vol}_n(D) = \left|\det \begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix}\right|$$

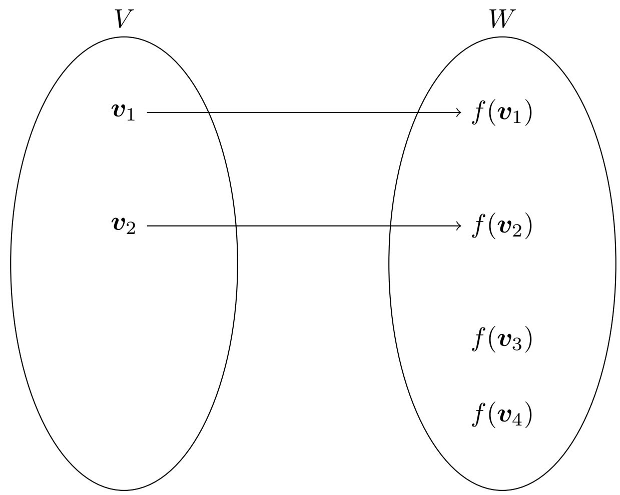
Volume of a Region Under a Linear Transformation
Let $D \subset \mathbb{R}^n$ and $T(\boldsymbol{x}) = A\boldsymbol{x}$ be a linear transformation. The volume of D, $\text{vol}_n(D)$, and the volume of the transformed region $T(D)$ are related by:

$$\text{vol}_n(T(D)) = |\det(A)|\,\text{vol}_n(D)$$

where $\det(A)$ is the determinant of A.

Injective
Let $f : V \to W$ be a linear map. We say that f is injective or one-to-one (or sometimes, f is an injection) if the following holds: For all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, if $f(\boldsymbol{v}_1) = f(\boldsymbol{v}_2)$, then $\boldsymbol{v}_1 = \boldsymbol{v}_2$.

That is, a map f is injective if any element in the codomain of f is the image of at most one element in its domain.



(Non-linear) Change of Variables

The Jacobian

Let $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ be a multivariable function defined by $f^i : A \subset \mathbb{R}^m \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \begin{bmatrix} f^1(\boldsymbol{x}) \\ \vdots \\ f^n(\boldsymbol{x}) \end{bmatrix}.$$

The Jacobian matrix of f at $\boldsymbol{x}_0$ is

$$[J_f(\boldsymbol{x}_0)] = \begin{bmatrix} D_1 f^1(\boldsymbol{x}_0) & D_2 f^1(\boldsymbol{x}_0) & \cdots & D_m f^1(\boldsymbol{x}_0) \\ D_1 f^2(\boldsymbol{x}_0) & D_2 f^2(\boldsymbol{x}_0) & \cdots & D_m f^2(\boldsymbol{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f^n(\boldsymbol{x}_0) & D_2 f^n(\boldsymbol{x}_0) & \cdots & D_m f^n(\boldsymbol{x}_0) \end{bmatrix},$$

if the partial derivatives exist.

Determinant of the Jacobian

Given a differentiable map $G(u, v) = (x(u, v), y(u, v))$, the Jacobian matrix, denoted as $[J_G]$, is the matrix of partial derivatives:

$$[J_G] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

The determinant of the Jacobian matrix is denoted as $\text{Jac}(G)$. Thus,

$$\det([J_G]) = \text{Jac}(G) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Approximation of the Volume of a Non-Linear Map

Let $D \subset \mathbb{R}^n$ be a region such that $\text{vol}_n(D)$ is small, and let $p \in D$. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. Then

$$\text{vol}_n(G(D)) \approx |\det([J_G](p))| \, \text{vol}_n(D).$$

That is, the n-dimensional volume of $G(D)$ can be approximated by the n-dimensional volume of $[J_G](p)(D)$.

Change of Variables
Let $K \subset \mathbb{R}^n$ be a compact set such that $\text{vol}_n(\partial K) = 0$. Let $U \subset \mathbb{R}^n$ be an open set containing K. Let $G : U \to \mathbb{R}^n$ be a map such that:

1. G is differentiable.
2. G is injective on the interior of K.
3. $\det([J_G]) \neq 0$ on the interior of K.

Then, if $f : G(K) \to \mathbb{R}$ is a continuous function, then

$$\int_{G(K)} f\, dV = \int_K (f \circ G)\, |\det([J_G])|\, dV.$$

Sometimes, it is easier to consider a map in the reverse direction, denoted as

$$F(\boldsymbol{x}, \boldsymbol{y}) = (u(x, y), v(x, y)).$$

Then let $G = F^{-1}$. If $G = F^{-1}$ and $\det([J_G]) \neq 0$, then

$$\det([J_G]) = \frac{1}{\det([J_F])}.$$

Example of Change of Variables - Volume of a Unit Sphere in $\mathbb{R}^3$
Consider the volume of the unit sphere in $\mathbb{R}^3$. The spherical coordinates transformation F maps from spherical coordinates (ρ, θ, ϕ) to Cartesian coordinates (x, y, z) and is given by:

$$F(\rho, \theta, \phi) = \begin{bmatrix} \rho \sin(\phi) \cos(\theta) \\ \rho \sin(\phi) \sin(\theta) \\ \rho \cos(\phi) \end{bmatrix}.$$

The Jacobian matrix $[J_F]$ of this transformation is:

$$[J_F] = \begin{bmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\theta)\sin(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{bmatrix}$$

$$\begin{aligned}
\det = {} & \cos(\phi) \cdot \det \begin{bmatrix} -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \rho\sin(\phi)\cos(\theta) & \rho\sin(\theta)\cos(\phi) \end{bmatrix} \\
& - 0 \cdot \det \begin{bmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\theta)\cos(\phi) \end{bmatrix} \\
& + (-\rho\sin(\phi)) \cdot \det \begin{bmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \end{bmatrix} \\
= {} & \cos(\phi) \cdot ((-\rho\sin(\phi)\sin(\theta)) \cdot (\rho\sin(\theta)\cos(\phi)) - (\rho\cos(\phi)\cos(\theta)) \cdot (\rho\sin(\phi)\cos(\theta))) \\
& + \rho\sin(\phi) \cdot ((\sin(\phi)\cos(\theta)) \cdot (\rho\sin(\phi)\cos(\theta)) - (-\rho\sin(\phi)\sin(\theta)) \cdot (\sin(\phi)\sin(\theta))) \\
= {} & \rho^2\cos(\phi)\sin(\phi)\sin^2(\theta) - \rho^2\cos^2(\phi)\cos(\theta)\sin(\phi) \\
& + \rho^2\sin(\phi)\cos(\theta)\sin^2(\phi) + \rho^2\sin^3(\phi)\sin(\theta)\cos(\theta) \\
= {} & \rho^2\sin(\phi)(\cos(\phi)\sin^2(\theta) - \cos^2(\phi)\cos(\theta) + \sin^2(\phi)\cos(\theta) + \sin^2(\theta)\cos(\theta)\sin(\phi)) \\
= {} & \rho^2\sin(\phi)(\cos(\phi) - \cos^2(\phi) + \sin^2(\phi)) \\
= {} & -\rho^2\sin(\phi).
\end{aligned}$$

The determinant of the Jacobian matrix $\det([J_F])$ is:

$$\det([J_F]) = -\rho^2 \sin(\theta).$$

And we consider:

$$|\det([J_F])| = \rho^2 \sin(\theta).$$

To find the volume of the unit sphere, integrate $\det([J_F])$ over the appropriate bounds for ρ, θ, and ϕ:

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin(\theta)\, d\rho\, d\theta\, d\phi.$$

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 \sin(\theta)\, d\rho\, d\theta\, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{3}\rho^3\right]_0^1 \sin(\theta)\, d\theta\, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} \sin(\theta)\, d\theta\, d\phi \\
&= \int_0^{2\pi} \left[-\frac{1}{3}\cos(\theta)\right]_0^{\pi} d\phi \\
&= \int_0^{2\pi} \frac{2}{3}\, d\phi \\
&= \left[\frac{2}{3}\phi\right]_0^{2\pi} \\
&= \frac{4\pi}{3}.
\end{aligned}$$

Volume of the Unit Ball in $\mathbb{R}^4$ Using Spherical Coordinates

The spherical coordinate system in $\mathbb{R}^4$ extends the traditional system in $\mathbb{R}^3$ by introducing an additional angle, leading to coordinates (r, ψ, θ, ϕ). In this system, a point in $\mathbb{R}^4$ is represented as:

$$\begin{aligned}
x &= r \sin(\psi) \sin(\theta) \cos(\phi), \\
y &= r \sin(\psi) \sin(\theta) \sin(\phi), \\
z &= r \sin(\psi) \cos(\theta), \\
w &= r \cos(\psi),
\end{aligned}$$

where $0 \le r \le 1$, $0 \le \psi \le \pi$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

The volume of the unit ball in $\mathbb{R}^4$ is computed using the integral:

$$\text{Volume} = \int_0^1 \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \left|-r^3 \sin^2(\psi) \sin(\theta)\right| d\phi\, d\theta\, d\psi\, dr.$$

Without delving into the specifics of the Jacobian determinant calculation, this setup directly leads to the volume of the unit ball in $\mathbb{R}^4$ being $\frac{\pi^2}{2}$.

Post Midterm 1 Material

1 Curves and Surfaces

Surjective
A map $f : X \to Y$ is **surjective** if for every $\boldsymbol{y} \in Y$, there exists an $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) = \boldsymbol{y}$.

Injective
A map $f : X \to Y$ is **injective** if $f(\boldsymbol{x_1}) = f(\boldsymbol{x_2}) \implies \boldsymbol{x_1} = \boldsymbol{x_2}$

1.1 Curves

Strict Parametrization
A strict parametrization of a curve $\mathcal{C} \subset \mathbb{R}^n$ is a vector-valued function $\boldsymbol{r}(t) : (a, b) \subset \mathbb{R} \to \mathcal{C}$ satisfying the following conditions:

1. $\boldsymbol{r}(t)$ surjects onto $\mathcal{C}$.
2. $\boldsymbol{r}(t)$ is injective for all $t \in (a, b)$.
3. $\boldsymbol{r}(t)$ is differentiable.
4. $\boldsymbol{r}'(t) \neq 0$ for all $t \in (a, b)$.

Arclength
Let C be a curve in $\mathbb{R}^n$, and let $\boldsymbol{r}(t) : (a, b) \to \mathbb{R}^n$ be a (strict) parametrization of C. Then the arclength of C is defined to be the integral

$$\int_a^b \|\boldsymbol{r}'(t)\| \, dt$$

Scalar Line Integral
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, and let $\mathcal{C}$ be a curve in $\mathbb{R}^n$. Let $\boldsymbol{r}(t) : (a, b) \to \mathbb{R}^n$ be a (strict) parametrization of C. Then the scalar line integral of f over $\mathcal{C}$ is denoted $\int_C f \, ds$, and is defined as the integral

$$\int_a^b f(\boldsymbol{r}(t)) \, \|\boldsymbol{r}'(t)\| \, dt$$

Open Subset
Let $A \subset \mathbb{R}^n$. We say that A is an open subset of $\mathbb{R}^n$ if A does not contain any of its boundary points. That is,

$$A \cap \partial A = \varnothing$$

Parameterization of a Curve (non-strict)
Let $\mathcal{C}$ be a curve in $\mathbb{R}^n$. Let $A \subset \mathbb{R}$ be a subset such that $\text{vol}_1(\partial A) = 0$. Let $X \subset A$ be a subset such that $A - X$ is open. Then $\gamma : A \to \mathbb{R}^n$ is a parametrization of $\mathcal{C}$ if:

1. $\mathcal{C} \subset \gamma(A)$ (that is, γ surjects onto $\mathcal{C}$).
2. $\gamma(A - X) \subset \mathcal{C}$, and $\gamma : A - X \to \mathcal{C}$ is injective.
3. $\gamma(t)$ is differentiable for all $t \in A - X$.
4. $\gamma'(t) \neq \mathbf{0}$ for all $t \in A - X$.
5. $\text{vol}_1(X) = 0$.

1.2 Surfaces

Strict Parameterization of a Surface
A strict parametrization of a surface $\mathcal{S} \subset \mathbb{R}^3$ is a multivariable function $G(u, v) : U \subset \mathbb{R}^2 \to \mathcal{S}$ satisfying the following conditions:

1. U is an open set.
2. $G(u, v)$ surjects onto $\mathcal{S}$.
3. $G(u, v)$ is injective for all $\boldsymbol{u} \in U$.
4. $G(u, v)$ is differentiable for all $\boldsymbol{u} \in U$ (that is, $\partial G/\partial u$ and $\partial G/\partial v$ exist).
5. The Jacobian matrix $[J_G(u, v)]$ is injective (i.e., has full rank) for all $\boldsymbol{u} \in U$.

Useful Statements on Injectivity
The following statements are equivalent about a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A:

1. T is injective.
2. The only solution to the equation $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.
3. If the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it is unique.
4. The columns of A are linearly independent.

Tangent Plane to Surface
The tangent plane to a surface S at a point $G(u_0, v_0)$ is spanned by the vectors

$$\frac{\partial G}{\partial u}(u_0, v_0) = \left(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\right)$$

and

$$\frac{\partial G}{\partial v}(u_0, v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0)\right).$$

Parameterization of a Surface (non-strict)
Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface. Let $A \subset \mathbb{R}^2$ be a subset such that $\text{vol}_2(\partial A) = 0$. Let $X \subset A$ be a subset such that $A - X$ is open. Then a map $\gamma : A \to \mathbb{R}^3$ parametrizes $\mathcal{S}$ if:

1. $\mathcal{S} \subset \gamma(A)$ (that is, γ surjects onto $\mathcal{S}$).
2. $\gamma(A - X) \subset \mathcal{S}$, and $\gamma : A - X \to \mathcal{S}$ is injective.
3. γ is differentiable for all $\boldsymbol{u} \in A - X$.
4. $[J_\gamma(\boldsymbol{u})]$ is injective for all $\boldsymbol{u} \in A - X$.
5. $\text{vol}_2(X) = 0$ and for any compact subset $\mathcal{C} \subset \mathcal{S}$, $\text{vol}_2(\gamma(X) \cap \mathcal{C}) = 0$.

k-Dimensional Volume Zero
Let $X \subset \mathbb{R}^n$ be a bounded subset. We say that X has k-dimensional volume 0 ($\text{vol}_k(X) = 0$) if

$$\lim_{N\to\infty} \sum_{\substack{C \in D_N(\mathbb{R}^n) \\ C \cap X \neq \varnothing}} \left(\frac{1}{2^N}\right)^k = 0.$$

Furthermore, now let $X \subset \mathbb{R}^n$ be an arbitrary subset. We say that X has k-dimensional volume 0 if for all $R > 0$, the intersection $B_R(\mathbf{0}) \cap X$ has volume 0, where $B_R(\mathbf{0})$ denotes the ball of radius R centered at the origin in $\mathbb{R}^n$.

Surface Area Integral
Let $\mathcal{S}$ be a surface (strictly) parametrized by a function $\gamma : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Then the surface area of $\mathcal{S}$ is given by

$$\int_{\mathcal{S}} d\mathcal{S} = \int_U \left\| \frac{\partial \gamma}{\partial u} \times \frac{\partial \gamma}{\partial v} \right\| du\, dv,$$

where $\frac{\partial \gamma}{\partial u}$ and $\frac{\partial \gamma}{\partial v}$ are the partial derivatives of γ with respect to u and v, respectively, and $\times$ denotes the cross product.

2 Manifolds

K-Dimensional Manifold as the Graph of a Function
A subset $\mathcal{M} \subset \mathbb{R}^n$ is a differentiable k-dimensional manifold embedded in $\mathbb{R}^n$ if, for all $\boldsymbol{x} \in \mathcal{M}$, there exists an open neighborhood $U \subset \mathbb{R}^n$ such that $\mathcal{M} \cap U$ is the graph of a C^1 mapping $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$.

Parameterization of a Manifold
Let $\mathcal{M} \subset \mathbb{R}^n$ be a k-dimensional manifold embedded in $\mathbb{R}^n$. Let $A \subset \mathbb{R}^k$ be a subset such that $\text{vol}_k(\partial A) = 0$. Let $X \subset A$ be a subset such that $A - X$ is open. Then a map $\gamma : A \to \mathbb{R}^n$ parametrizes $\mathcal{M}$ if:

(a) $\mathcal{M} \subset \gamma(A)$ (that is, γ surjects onto $\mathcal{M}$).

(b) $\gamma(A - X) \subset \mathcal{M}$, and $\gamma : A - X \to \mathcal{M}$ is injective.

(c) γ is differentiable for all $\boldsymbol{u} \in A - X$.

(d) $[J_\gamma(\boldsymbol{u})]$ is injective for all $\boldsymbol{u} \in A - X$.

(e) $\text{vol}_k(X) = 0$ and for any compact subset $\mathcal{C} \subset \mathcal{M}$, $\text{vol}_k(\gamma(X) \cap \mathcal{C}) = 0$.

Differentiable Manifold and Tangent Space
Let $M \subset \mathbb{R}^n$ be a differentiable k-dimensional manifold. Consider a neighborhood U of a point $\boldsymbol{z}_0 = (\boldsymbol{x}_0, \boldsymbol{y}_0) \in M$ such that the intersection of M and U can be represented as:

$$M \cap U = \left\{ (\boldsymbol{x}, f(\boldsymbol{x})) \mid \boldsymbol{x} \in \mathbb{R}^k \right\},$$

where $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is a differentiable function that locally describes M in the neighborhood of $\boldsymbol{z}_0$.

The **tangent space** to M at $\boldsymbol{z}_0$, denoted $T_{\boldsymbol{z}_0}M$, is defined as the graph of the derivative of f at $\boldsymbol{x}_0$, denoted $[J_f(\boldsymbol{x}_0)]$. This derivative, also known as the Jacobian matrix of f at $\boldsymbol{x}_0$, maps directions in the input space $\mathbb{R}^k$ to directions in the output space $\mathbb{R}^{n-k}$, effectively describing how the manifold M changes direction at the point $\boldsymbol{z}_0$. Mathematically, the tangent space can be expressed as:

$$T_{\boldsymbol{z}_0}M = \left\{ (\boldsymbol{x}, [J_f(\boldsymbol{x}_0)](\boldsymbol{x})) \mid \boldsymbol{x} \in \mathbb{R}^k \right\}.$$

The **tangent space** to a manifold described by a parameterization is defined as

$$T\gamma(\boldsymbol{u})M = \text{Im}[J\gamma(\boldsymbol{u})]$$

That is, the tangent space $T\gamma(\boldsymbol{u})M$ at $\boldsymbol{u}$ can be expressed as the image of the Jacobian matrix of φ at $\boldsymbol{u}$, which maps vectors from $\mathbb{R}^k$ into $\mathbb{R}^n$. Mathematically, this is represented as:

$$T\gamma(\boldsymbol{u})M = \left\{ [J\gamma(\boldsymbol{u})](\boldsymbol{x}) \in \mathbb{R}^n \mid \boldsymbol{x} \in \mathbb{R}^k \right\}.$$

Volume of a k-Dimensional Parallelepiped in $\mathbb{R}^k$
Let D be the k-dimensional parallelepiped spanned by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^k$. Consider the $k \times k$ matrix T given by the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as columns. Then, the volume of D is given by

$$\text{volume}(D) = |\det(T)| = \sqrt{\det(T^\top T)},$$

where $T^\top$ denotes the transpose of T.

Volume of a k-Dimensional Parallelepiped in $\mathbb{R}^k$ and $\mathbb{R}^n$
Let D be the k-dimensional parallelepiped spanned by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^k$. Consider the $k \times k$ matrix T given by the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as columns. Then, the volume of D in $\mathbb{R}^k$ is given by

$$\text{volume}(D) = |\det(T)| = \sqrt{\det(T^\top T)},$$

where $T^\top$ denotes the transpose of T.

Furthermore, now let D be the k-dimensional parallelepiped spanned by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^n$. While the $\det(T)$ is meaningless in this context, we have

$$\text{volume}(D) = \sqrt{\det(T^\top T)},$$

meaning the k-dimensional volume in $\mathbb{R}^n$.

Integral Over a Manifold
Let $\mathcal{M} \subset \mathbb{R}^n$ be a differentiable k-dimensional manifold, let $A \subset \mathbb{R}^k$ be a set with well-defined volume, and let $\gamma : A \to \mathbb{R}^n$ be a parametrization of $\mathcal{M}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say f is integrable over $\mathcal{M}$ if the following integral exists and is well-defined:

$$\int_{\mathcal{M}} f \, d\mathcal{M} = \int_A f(\gamma(\boldsymbol{u})) \sqrt{\det\left([J_\gamma(\boldsymbol{u})]^\top [J_\gamma(\boldsymbol{u})]\right)} \, d\boldsymbol{u},$$

Useful Statements on Surjectivity
The following statements are equivalent about a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A:

(a) T is surjective.

(b) The columns of A span $\mathbb{R}^m$.

(c) For every $\boldsymbol{b} \in \mathbb{R}^m$, there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \boldsymbol{b}$.

(d) The rows of A are linearly indepedent.

2.1 Manifolds as Vanishing Loci

Vanishing Locus of a Function
Let $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. The vanishing locus of f (sometimes called the locus, or the zero locus) is the set of points $V(f)$ where f vanishes. That is,

$$V(f) = \{\boldsymbol{x} \in X \mid f(\boldsymbol{x}) = 0\}.$$

Locally showing a vanishing locus is a differentiable manifold

Let M be a subset of $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open, and let $F : U \to \mathbb{R}^{n-k}$ be a C^1-mapping such that

$$M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$$

If the derivative $[J_F(\boldsymbol{z})]$ is a surjective map for every $\boldsymbol{z} \in M \cap U$, then $X \cap U$ is a differentiable k-dimensional manifold embedded in $\mathbb{R}^n$.

Showing a vanishing locus is a differentiable manifold
Let M be a subset of $\mathbb{R}^n$. If for every $\boldsymbol{z} \in M$, there exists an open set $U \subset \mathbb{R}^n$ containing $\boldsymbol{z}$, and a C^1-mapping $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that

$$M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$$

and $[J_F(\boldsymbol{z})]$ is surjective for every $\boldsymbol{z} \in M$, then M is a differentiable k-dimensional manifold.

A differentiable manifold is locally a vanishing locus
Let $M \subset \mathbb{R}^n$ be differentiable k-dimensional manifold. Then every point $\boldsymbol{z} \in M$ has a neighborhood $U \subset \mathbb{R}^n$ such that there exists a C^1-mapping $F : U \to \mathbb{R}^{n-k}$ such that $[J_F(\boldsymbol{z})]$ is surjective, and

$$M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$$

Inverse Image of a Manifold Theorem
Let $M \subset \mathbb{R}^m$ be a differentiable k-dimensional manifold embedded in $\mathbb{R}^m$. Let $U \subset \mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a C^1-mapping. Define $f^{-1}(M)$ to be the inverse image of M,

$$f^{-1}(M) = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \in M\}$$

If the derivative $[J_f(\boldsymbol{x})]$ is a surjective map for every $\boldsymbol{x} \in f^{-1}(M)$ in $\mathbb{R}^n$, then $f^{-1}(M)$ is a differentiable $k + n - m$-dimensional manifold embedded in $\mathbb{R}^n$.

Independence of Coordinates Corollary
Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of the form

$$g(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{c}$$

where $A \in M_{n\times n}(\mathbb{R})$ is an invertible $n \times n$ matrix. If M is a differentiable k-dimensional manifold, then $g(M)$ is also a differentiable k-dimensional manifold.

3 Vector Fields

Conservative Vector Field
A vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called conservative if there exists a differentiable function $f(x_1, \ldots, x_n)$ such that

$$\boldsymbol{F} = \nabla f = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle.$$

The function f is called a potential function for $\boldsymbol{F}$.

Divergence
Given a vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\boldsymbol{F}(\boldsymbol{u}) = \langle F_1(\boldsymbol{u}), \ldots, F_n(\boldsymbol{u})\rangle$, the divergence of $\boldsymbol{F}$ is the scalar-valued function $\text{div}\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\text{div}\boldsymbol{F}(\boldsymbol{u}) = \frac{\partial F_1}{\partial x_1}(\boldsymbol{u}) + \cdots + \frac{\partial F_n}{\partial x_n}(\boldsymbol{u}).$$

In operator notation, this is written as

$$\text{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \cdot \boldsymbol{F}.$$

The divergence of a vector field at a point P measures the net flux of $\boldsymbol{F}$ out of an infinitesimally small sphere centered at P. It characterizes the behavior of the vector field at P as follows:

- If $\text{div}\boldsymbol{F}(P) > 0$, then P is a source.
- If $\text{div}\boldsymbol{F}(P) < 0$, then P is a sink.
- If $\text{div}\boldsymbol{F}(P) = 0$, then P is said to be incompressible.

Curl
Given a vector field in $\mathbb{R}^3$, $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$, the curl of $\boldsymbol{F}$ is the vector field defined by

$$\text{curl}\boldsymbol{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

In operator notation, this can be written as

$$\text{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \boldsymbol{F}.$$

Orientation of a Curve
Given a curve $\mathcal{C}$, a continuous choice of tangent vector on $\mathcal{C}$ is called an orientation. A curve with a chosen orientation is called an oriented curve. Moving along the chosen direction is called the positive direction along $\mathcal{C}$, and moving against the chosen orientation is called the negative direction (along $\mathcal{C}$.

Given an oriented curve $\mathcal{C}$ in $\mathbb{R}^2$, we say that the positive direction across $\mathcal{C}$ is the direction that goes left to right from the perspective of the positive orientation along $\mathcal{C}$. Let $\boldsymbol{n}(p)$ denote the unit vector normal to $\mathcal{C}$ at the point p, pointing in the positive direction across $\mathcal{C}$.

Vector Line Integral
The line integral of a vector field $\boldsymbol{F}$ along an oriented curve $\mathcal{C}$ is denoted

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

We define it as the integral of the tangential component of $\boldsymbol{F}$ over $\mathcal{C}$. Formally,

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} := \int_{\mathcal{C}} (\boldsymbol{F} \cdot \boldsymbol{T})\, ds$$

where $\boldsymbol{T}$ is the unit tangent vector to $\mathcal{C}$, and ds represents a differential element of arc length along $\mathcal{C}$.

Let $\boldsymbol{r}(t)$ be a positively oriented regular parametrization of an oriented curve $\mathcal{C}$ for $a \leq t \leq b$. Then the line integral of $\boldsymbol{F}$ along $\mathcal{C}$ can be computed as

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt.$$

If $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$, then another common notation for line integrals is

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz.$$

Properties of Vector Line Integrals
Let $\mathcal{C}$ be a smooth oriented curve, and let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields.

1. **Linearity:**

- The line integral is linear with respect to vector fields:

$$\int_{\mathcal{C}} (\boldsymbol{F} + \boldsymbol{G}) \cdot d\boldsymbol{r} = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{\mathcal{C}} \boldsymbol{G} \cdot d\boldsymbol{r}.$$

- The line integral respects scalar multiplication:

$$\int_{\mathcal{C}} c\boldsymbol{F} \cdot d\boldsymbol{r} = c \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

2. Additivity:

- If $\mathcal{C}$ is the union of smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, then

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \cdots + \int_{\mathcal{C}_n} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

3. Reversing Orientation:

- If the orientation of $\mathcal{C}$ is reversed, denoted as $-\mathcal{C}$, then

$$\int_{-\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Fundamental Theorem of Conservative Vector Fields
Let $\boldsymbol{F} = \nabla f$ be a conservative vector field on a domain D. If $\boldsymbol{r}$ is a path along a curve $\mathcal{C}$ from point P to Q in D, then

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = f(Q) - f(P).$$

In particular, this implies that $\boldsymbol{F}$ is path-independent.

Corollary: Let $\boldsymbol{F} = \nabla f$ be a conservative vector field on a domain D. If $\boldsymbol{r}$ is a path along a closed curve $\mathcal{C}$ in D, then the circulation is zero:

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = 0.$$

Simply Connected
A simply connected domain is a path-connected domain where one can continuously shrink any simple closed curve into a point while remaining within the domain. For two-dimensional regions, a simply connected domain is one without holes. For three-dimensional domains, the concept of simply connected is more subtle; it refers to a domain without any holes going all the way through it.

From Zero Curl to Conservative
Let $\boldsymbol{F}$ be a vector field on a simply-connected domain D. If $\boldsymbol{F}$ satisfies the cross-partials condition (that is, the curl of $\boldsymbol{F}$ is zero), then $\boldsymbol{F}$ is conservative.

Path Independence
A vector field $\boldsymbol{F}$ on a domain D is path-independent if for any two points $P, Q \in D$, then

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

for any two paths $\mathcal{C}_1, \mathcal{C}_2$ in D that start at P and end at Q.

Normal Vector to Curve

Let $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ be a positively oriented parametrization of an oriented curve $\mathcal{C}$. Observe that $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$ is normal to $\mathcal{C}$. Therefore, the unit normal vector $\boldsymbol{n}(t)$ at any point on $\mathcal{C}$ is given by

$$\boldsymbol{n}(t) = \frac{\boldsymbol{N}(t)}{\|\boldsymbol{N}(t)\|}.$$

Vector Flux Integral

The flux integral of a vector field $\boldsymbol{F}$ along an oriented curve $\mathcal{C}$ in $\mathbb{R}^2$ is the integral of the normal component of $\boldsymbol{F}$:

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot \boldsymbol{n}\, ds.$$

Let $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ be a positively oriented parametrization of an oriented curve $\mathcal{C}$ for $a \le t \le b$. Then the flux integral of $\boldsymbol{F}$ along $\mathcal{C}$ can be computed as

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{N}(t)\, dt.$$

Surface Orientation

Given a surface $\mathcal{S} \subset \mathbb{R}^3$, a continuous choice of unit normal vector on $\mathcal{S}$ is called an orientation. A surface with a chosen orientation is called an oriented surface.

Recall that given a parametrization $\boldsymbol{G}(u, v)$ of $\mathcal{S}$, then the normal vector at a point $P = \boldsymbol{G}(u_0, v_0)$ on $\mathcal{S}$ is determined by

$$\boldsymbol{N}(P) = \frac{\partial \boldsymbol{G}}{\partial u}(u_0, v_0) \times \frac{\partial \boldsymbol{G}}{\partial v}(u_0, v_0).$$

Given an oriented surface, we say that a parametrization is positively oriented if the orientation given by

$$\frac{\boldsymbol{N}(P)}{\|\boldsymbol{N}(P)\|}$$

agrees with the orientation of $\mathcal{S}$.

If $\boldsymbol{G}(u, v)$ is a strict parametrization of $\mathcal{S}$, then the Jacobian matrix $[J_{\boldsymbol{G}}(u, v)]$ is injective. Hence, $\frac{\partial \boldsymbol{G}}{\partial u}(u_0, v_0)$ and $\frac{\partial \boldsymbol{G}}{\partial v}(u_0, v_0)$ are linearly independent, so $\boldsymbol{N}(P) \neq \boldsymbol{0}$. Otherwise, we have to worry about singularities in $\mathcal{S}$.

Vector Surface Integral

The vector surface integral of $\boldsymbol{F}$ over $\mathcal{S}$ is defined as

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\mathcal{S} := \iint_{\mathcal{S}} (\boldsymbol{F} \cdot \boldsymbol{n})\, d\mathcal{S}.$$

This is also known as the flux of $\boldsymbol{F}$ across (or through) $\mathcal{S}$.

Let $\boldsymbol{G}(u, v) : A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be an oriented parametrization of a surface $\mathcal{S}$. Then the vector surface integral can be computed as

$$\iint_{\mathcal{S}} (\boldsymbol{F} \cdot \boldsymbol{n})\, d\mathcal{S} = \iint_{A-X} \boldsymbol{F}(\boldsymbol{G}(u, v)) \cdot \boldsymbol{N}(u, v)\, du\, dv,$$

where $\boldsymbol{N}(u, v)$ is the normal vector at the point (u, v) on the parametrization domain A, ensuring the orientation matches that of $\mathcal{S}$.

Flipped Orientation

If $-\mathcal{S}$ denotes $\mathcal{S}$ with the opposite orientation, then the vector surface integral with the flipped orientation is given by

$$\iint_{-\mathcal{S}} (\boldsymbol{F} \cdot \boldsymbol{n})\, d\mathcal{S} = -\iint_{\mathcal{S}} (\boldsymbol{F} \cdot \boldsymbol{n})\, d\mathcal{S}.$$

Simple Closed Curve
A simple closed curve $\mathcal{C}$ is a closed curve that does not intersect itself.
Note: A simple closed curve $\mathcal{C}$ in $\mathbb{R}^3$ can be thought of as the boundary of a surface $\mathcal{S}$ in $\mathbb{R}^3$.

Jordan Curve Theorem
A simple closed curve $\mathcal{C}$ in $\mathbb{R}^2$ splits $\mathbb{R}^2$ into exactly two regions: an interior region D, and the exterior region $\mathbb{R}^2 - D$.

4 Green's theorem, Stokes' theorem, and the Divergence theorem

Green's Theorem
Let D be a region in $\mathbb{R}^2$ such that ∂D is a disjoint union of simple closed curves, with ∂D oriented so that D is always to the left. Suppose $\boldsymbol{F} = \langle F_1, F_2 \rangle$ is a smooth vector field on D. Then

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

Green's Theorem in Circulation Form
Let D be a region in $\mathbb{R}^2$ such that ∂D is a simple closed curve, oriented counterclockwise. Suppose $\boldsymbol{F} = \langle F_1, F_2 \rangle$ is a smooth vector field on D. Then

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \operatorname{curl}_z(\boldsymbol{F})\, dA.$$

Green's Theorem in Flux Form
Let D be a region in $\mathbb{R}^2$ such that ∂D is a simple closed curve, oriented counterclockwise. Suppose $\boldsymbol{F} = \langle F_1, F_2 \rangle$ is a smooth vector field on D. Then

$$\oint_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \iint_D \operatorname{div}(\boldsymbol{F})\, dA.$$

Additivity of Circulation
Let D be a region in $\mathbb{R}^2$ such that ∂D is a simple closed curve, oriented counterclockwise. If we decompose a domain D into two domains D_1 and D_2 which intersect only on their boundaries, ∂D_1 and ∂D_2, then

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\partial D_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{\partial D_2} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Upper Half Space
The upper half-space $H_k \subset \mathbb{R}^k$ is the (closed) set

$$H_k := \{\boldsymbol{x} = \langle x_1, \ldots, x_k \rangle \mid x_k \geq 0\}.$$

This is a k-dimensional manifold with boundary

$$\partial H_k = \{\langle x_1, \ldots, x_k \rangle \mid x_k = 0\}.$$

Manifold with Boundary
A subset $\mathcal{M} \subset \mathbb{R}^n$ is a differentiable k-dimensional manifold with boundary embedded in $\mathbb{R}^n$ if for all $\boldsymbol{z} \in \mathcal{M}$, either:

1. There exists an open neighborhood $U \subset \mathbb{R}^n$ such that there exists a C^1-mapping $F: U \to \mathbb{R}^{n-k}$ such that
   - $\mathcal{M} \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$
   - $[J_F(\boldsymbol{z})]$ is surjective.
2. Or, there exists an open neighborhood $V \subset \mathbb{R}^n$ such that there exists a C^1-mapping $G: V \to \mathbb{R}^{m+n-k}$ such that
   - $G(\boldsymbol{x}) = \langle F_1(\boldsymbol{x}), F_2(\boldsymbol{x})\rangle$
   - $F_1 : V \to \mathbb{R}^{n-k}$, and $F_2 : V \to \mathbb{R}^m$
   - $G(\boldsymbol{z}) = \boldsymbol{0}$
   - $\mathcal{M} \cap V = \{\boldsymbol{x} \in V \mid F_1(\boldsymbol{x}) = \boldsymbol{0}, F_2(\boldsymbol{x}) \geq 0\}$
   - $[J_G(\boldsymbol{z})]$ is surjective.

We say that the set of points $\boldsymbol{z} \in \mathcal{M}$ satisfying the latter condition are the boundary of $\mathcal{M}$.

If $\boldsymbol{z} \in \partial\mathcal{M}$ satisfies the latter condition, we say that $\boldsymbol{z}$ is a corner point of codimension m. In the special case $m = 1$, then we say that $\boldsymbol{z}$ is in the smooth boundary of $\mathcal{M}$ (denoted $\partial_s\mathcal{M}$). The set of corner points that is not in $\partial_s\mathcal{M}$ is called the non-smooth boundary of $\mathcal{M}$.

Boundary Orientation

Recall that an orientation of a surface $\mathcal{S}$ in $\mathbb{R}^3$ is a (continuous) choice of a unit normal vector $\boldsymbol{n}(\boldsymbol{P})$ at each point $\boldsymbol{P}$ on $\mathcal{S}$. If $\mathcal{S}$ is an oriented surface, then we can specify an orientation of the boundary $\partial\mathcal{S}$.

The boundary orientation of $\partial\mathcal{S}$ is chosen so that if your feet are on $\mathcal{S}$, and your head is where the head of $\boldsymbol{n}(\boldsymbol{P})$ is, then the orientation of $\partial\mathcal{S}$ is chosen so that $\mathcal{S}$ is always to your left.

Stoke's Theorem

Let $\boldsymbol{G}(u, v) : D \to \mathbb{R}^3$ be a positively oriented parametrization of a surface $\mathcal{S}$. This determines an orientation on $\partial\mathcal{S}$ as described previously. Suppose $\boldsymbol{F}$ is a smooth vector field on a solid region W containing $\mathcal{S}$. Then

$$\oint_{\partial\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{\mathcal{S}} \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n}\, d\mathcal{S},$$

where $\boldsymbol{n}$ is the unit normal vector to $\mathcal{S}$, chosen according to the orientation of $\mathcal{S}$.

Corollary of Stoke's Theorem: Interpreting Curl

Suppose $\boldsymbol{F}$ is a vector field in $\mathbb{R}^3$, and consider a plane through a point $X \in \mathbb{R}^3$ with unit normal vector $\boldsymbol{n}$. Let C be a small circle of radius ϵ in the plane, centered at P, which encloses a disk D in the plane. Then

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n}\, dS \approx (\operatorname{curl}(\boldsymbol{F})(P) \cdot \boldsymbol{n})\operatorname{area}(D).$$

Thus,

$$(\operatorname{curl}(\boldsymbol{F})(P) \cdot \boldsymbol{n}) \approx \frac{1}{\operatorname{area}(D)} \oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Therefore, the circulation of $\boldsymbol{F}$ in a given plane depends on the angle between curl($\boldsymbol{F}$) and $\boldsymbol{n}$.

Closed Surface

A closed surface is a surface with boundary (i.e., a 2-dimensional manifold with boundary) that has no boundary. That is, $\partial\mathcal{S} = \emptyset$.

Corollary: Let $\mathcal{S}$ be a closed surface. Then

$$\iint_{\mathcal{S}} \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n}\, d\mathcal{S} = 0.$$

Vector Potential

Let $\boldsymbol{F}$ be a vector field defined on a region $W \subset \mathbb{R}^3$. Suppose

$$\boldsymbol{F} = \text{curl}(\boldsymbol{A})$$

for some vector field $\boldsymbol{A}$. Then $\boldsymbol{A}$ is called a vector potential for $\boldsymbol{F}$ on W.

Warning: Vector potentials are not unique.

Theorem from Stoke's & Vector Potentials

If $\boldsymbol{A}$ is a vector potential for $\boldsymbol{F}$ on W, then under the conditions of Stoke's theorem,

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot \boldsymbol{n} \, d\mathcal{S} = \iint_{\mathcal{S}} \text{curl}(\boldsymbol{A}) \cdot \boldsymbol{n} \, d\mathcal{S} = \oint_{\partial \mathcal{S}} \boldsymbol{A} \cdot d\boldsymbol{r}.$$

In other words, the surface integral of $\boldsymbol{F} = \text{curl}(\boldsymbol{A})$ is surface-independent.

Corollary: If $\boldsymbol{F}$ has a vector potential $\boldsymbol{A}$ on W, and $\mathcal{S}$ is a closed surface in W, then

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot \boldsymbol{n} \, d\mathcal{S} = 0.$$

Divergence Theorem

Let $\mathcal{S}$ be a closed surface that encloses a region $W \subset \mathbb{R}^3$, such that $\mathcal{S}$ is piecewise smooth, and is oriented by normal vectors pointing away from W.

If $\boldsymbol{F}$ is a smooth vector field defined on an open region containing W, then

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot \boldsymbol{n} \, d\mathcal{S} = \iiint_{W} \text{div}(\boldsymbol{F}) \, dV.$$