Math 184 Running Notes

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Counting Principles

Definition (Injection)**.** A function $f : A \to B$ is an **injection** if for all $a, a' \in A$, $a \neq a'$ implies $f(a) \neq f(a')$.

Definition (Surjection)**.** A function $f : A \to B$ is a **surjection** if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition (Bijection)**.** A function $f : A \to B$ is a **bijection** if it is both injective and surjective.

Definition (Inverse Function)**.** A function $g : B \to A$ is called the **inverse** of $f : A \to B$ if

$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B$$

In this case, $g = f^{-1}$.

Theorem. A function $f : A \to B$ is a bijection if and only if it has an inverse.

Definition (Cardinality)**.** The **cardinality** of a finite set A is the number of elements in A.

Proposition (Cardinality of Equivalent Sets)**.** If $A \cong B$ (i.e., there exists a bijection $A \to B$), then $|A| = |B|$.

Definition (Disjoint Sets)**.** Sets A and B are **disjoint** if $A \cap B = \emptyset$.

Proposition (Addition Principle)**.** If A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

General version: If $A_1, \ldots, A_n$ are disjoint, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

Problem 1. How many squares are in the figure below?

$$\text{Let } S = \{\text{all squares in the figure}\}, \quad S_k = \{k \times k \text{ squares}\}$$

Then,

$$S = S_1 \cup S_2 \cup S_3$$

By the Addition Principle,

$$|S| = |S_1| + |S_2| + |S_3| = 9 + 4 + 1 = 14$$

Definition (Cartesian Product)**.** The **Cartesian product** of sets A and B is defined as

$$A \times B := \{(a, b) : a \in A,\, b \in B\}$$

Proposition (Multiplication Principle)**.**

$$|A \times B| = |A| \cdot |B|$$

More generally, for sets $A_1, \ldots, A_n$,

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$

Notation 1.

$$[n] := \{1, 2, \ldots, n\}$$

Definition (Permutation)**.** A **permutation** of a set A is a linear ordering of the elements of A, or equivalently, a bijection

$$f : [n] \to A \quad \text{where } n := |A|$$

Let S_A denote the set of permutations of A.

Notation 2. The factorial of n is defined as

$$n! := n(n-1)(n-2)\cdots 1$$

Example: $3! = 3 \cdot 2 \cdot 1 = 6$

Theorem. If A is a finite set with $|A| = n$, then the number of permutations of A is

$$|S_A| = n!$$

Definition (Partial Permutation)**.** Let A be an n-element set. A **partial permutation** of A is a linear ordering of k elements of A.

Equivalently, partial permutations correspond to injections

$$f : [k] \hookrightarrow A$$

Theorem. The number of injections $[k] \hookrightarrow A$ is

$$n(n-1)(n-2)\cdots(n-k+1) =: (n)_k$$

where $(n)_k$ is the **falling factorial**.

Proposition (Subtraction Principle)**.** Let $A \subseteq B$. Then the size of the set difference is

$$|B \setminus A| = |B| - |A|$$

Proposition (Division Principle, Version 1)**.** Suppose $f : A \twoheadrightarrow B$ is a d**-to-1** map. That is,

$$\forall b \in B, \quad |f^{-1}(b)| = d \quad \text{where } f^{-1}(b) = \{a \in A : f(a) = b\}$$

Then,

$$|B| = \frac{|A|}{d}$$

Definition (Circular Permutation)**.** A **circular permutation** of a set A is an arrangement of the elements of A around a circle such that rotations of the same arrangement are considered the same (but not reflections).

Definition (Alphabet and Word)**.** An **alphabet** A is a finite set whose elements are called **letters**.

A **word** in A is a sequence of letters from A (including the **empty word**).

The number of letters in a word is called the **length** of the word.

Proposition. The number of words of length k over an n-letter alphabet is

$$n^k$$

Proposition. The number of subsets of an n-element set A is 2^n. That is,

$$|\mathcal{P}(A)| = 2^{|A|}$$

Notation 3. Let

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Proposition. The number of words of length n over the alphabet $A = \{0, 1\}$ that contain exactly k 1's (and $n-k$ 0's) is

$$\binom{n}{k}$$

Definition (Lattice Path)**.** A **lattice path** L from $(0,0)$ to (m,n) is a sequence $(v_0, v_1, \ldots, v_k)$ such that

$$v_0 = (0,0), \quad v_k = (m,n), \quad \text{and} \quad v_{i+1} - v_i \in \{(1,0),(0,1)\} \quad \text{for all } i = 0, \ldots, k-1.$$

Proposition. The number of lattice paths from $(0,0)$ to (m,n) is

$$\binom{m+n}{n}$$

Proposition.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof.

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

This identity follows from the binomial expansion of $(1+1)^n$. □

Proposition (Pascal's Recurrence)**.**

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's Triangle:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Proposition (Binomial Convolution Identity).

$$\binom{\ell+m}{n} = \sum_{k=0}^{n} \binom{\ell}{k}\binom{m}{n-k}$$

Proof. The left-hand side counts the number of ways to choose n elements from $A \cup B$ where $|A| = \ell$ and $|B| = m$.

The right-hand side breaks this down by choosing k elements from A and $n-k$ from B, summing over all possible k. □

Theorem (Binomial Theorem). For every $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof. By induction on n. □

Definition (Weak Composition). A **weak composition** of k into n parts is a solution to the equation

$$x_1 + x_2 + \cdots + x_n = k \quad \text{where } x_i \in \mathbb{Z},\ x_i \geq 0$$

Theorem (Weak Composition Theorem). For any integers $n \geq 1$ and $k \geq 0$, the number of weak compositions of k into n parts — that is, the number of nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_n = k$$

— is

$$\binom{n+k-1}{k}.$$

Example. Let $n = 3$ and $k = 2$. Then there are 6 weak compositions:

$$0+0+2$$
$$0+2+0$$
$$2+0+0$$
$$1+1+0$$
$$1+0+1$$
$$0+1+1$$

There are 6 weak compositions.

Definition (Multiset). A **multiset** is a generalization of a set in which elements may be repeated.

Notation 4. Let

$$\left(\!\!\binom{n}{k}\!\!\right) := \binom{n+k-1}{k}$$

denote the number of multisets of size k from a set of n elements.

Theorem. The number of multisets of size k from a set S is

$$\left(\!\!\binom{|S|}{k}\!\!\right) = \binom{|S|+k-1}{k}$$

Proof. We count the number of weak compositions of k into n parts.

We imagine placing $n-1$ bars among k indistinguishable stars to divide them into n chambers. This corresponds to choosing $n-1$ positions for bars among $k+n-1$ total slots.

Hence the number of such compositions is

$$\binom{n+k-1}{k}$$

□

Definition (Composition)**.** A **composition** of k into n parts is a solution to the equation

$$x_1 + x_2 + \cdots + x_n = k \quad \text{where } x_i \in \mathbb{Z},\ x_i \geq 1$$

Theorem. The number of compositions of k into n parts is

$$\binom{k-1}{n-1}$$

Notation 5. Let

$$\binom{n}{a_1, \ldots, a_k} := \frac{n!}{a_1! \cdots a_k!} \quad \text{where } a_1 + \cdots + a_k = n$$

Theorem. Let

$$M = \{a_1 \cdot x_1,\ a_2 \cdot x_2,\ \ldots,\ a_k \cdot x_k\}$$

be a multiset with $n = a_1 + \cdots + a_k$ elements. Then the number of distinct permutations of M is

$$|S_M| = \binom{n}{a_1, \ldots, a_k}$$

Theorem (Multinomial Theorem)**.**

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{a_1 + \cdots + a_k = n} \binom{n}{a_1, \ldots, a_k} x_1^{a_1} \cdots x_k^{a_k}$$

Definition (Generalized Binomial Coefficient)**.** We define the **generalized binomial coefficient** for any $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$ by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

Theorem (Newton's Binomial Theorem)**.**

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{whenever the LHS makes sense}$$

Theorem.

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\binom{n+k-1}{k}\!\right) x^k \quad \text{(a generating function using generalized binomial coefficients)}$$

Proof. Use Newton's Binomial Theorem with $\alpha = -n$:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

Using the identity

$$\binom{-n}{k} = (-1)^k \left(\!\binom{n+k-1}{k}\!\right)$$

we get

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \left(\!\binom{n+k-1}{k}\!\right) x^k = \sum_{k=0}^{\infty} \left(\!\binom{n+k-1}{k}\!\right) (-x)^k$$

Now replace x with $-x$:

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\binom{n+k-1}{k}\!\right) x^k$$

□

Example. For $n = 1$,

$$\left(\!\binom{1+k-1}{k}\!\right) = \binom{k}{k} = 1 \quad \text{for all } k \Rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Generating Functions

Definition (Generating Function)**. The generating function for (A, w) is the formal power series**

$$F(x) := \sum_{a \in A} x^{w(a)}, \quad \underline{\text{Note:}} \quad F(1) = |A|.$$

Lemma.

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Proof.

$$\begin{aligned}
F(x) &= \sum_{a \in A} x^{w(a)} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{a \in A \\ w(a)=n}} x^{w(a)} \\
&= \sum_{n=0}^{\infty} a_n x^n.
\end{aligned}$$

□

Definition ((Formal Power Series))**.** A **formal power series** is an expression of the form

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

We do not require any notions of convergence, and usually do not want to plug in values for x. Instead, we only care about the coefficients of $F(x)$.

Notation 6. We denote the coefficient of x^n in $F(x)$ by

$$[x^n]F(x)$$

Proposition ((Operations on Formal Power Series))**.** We can perform several familiar operations on formal power series:

1. **Addition**: If $F(x) = \sum_n a_n x^n$, $G(x) = \sum_n b_n x^n$, then

$$F(x) + G(x) := \sum_n (a_n + b_n) x^n$$

2. **Multiplication**:

$$F(x) \cdot G(x) = \sum_n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \qquad \text{(same as polynomials)}$$

3. **Differentiation**:

$$F'(x) := \sum_{n=0}^{\infty} n a_n x^{n-1}$$

4. **Integration**:

$$\int F(x)\, dx := \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Theorem ((Multiplication Principle for Generating Functions))**.** Let

$$S \overset{\sim}{\longleftrightarrow} A \times B \times C \times D \times \cdots$$

be a bijection, where $s \leftrightarrow (a, b, c, d, \ldots) \in S$.

Suppose there are weight functions:

$$\begin{gathered} w : S \to \mathbb{N}_{\geq 0} \\ \alpha : A \to \mathbb{N}_{\geq 0} \\ \beta : B \to \mathbb{N}_{\geq 0} \\ \gamma : C \to \mathbb{N}_{\geq 0} \\ \delta : D \to \mathbb{N}_{\geq 0} \\ \vdots \end{gathered}$$

such that

$$w(s) = \alpha(a) + \beta(b) + \gamma(c) + \delta(d) + \cdots .$$

Then the generating function satisfies

$$\sum_{s \in S} x^{w(s)} = \left(\sum_{a \in A} x^{\alpha(a)}\right)\left(\sum_{b \in B} x^{\beta(b)}\right)\left(\sum_{c \in C} x^{\gamma(c)}\right)\left(\sum_{d \in D} x^{\delta(d)}\right)\cdots$$

Example (Binomial Theorem again)**.** Let $\mathcal{S} = \{\text{binary strings of length } n\}$

$$A_1 = \cdots = A_n = \{0, 1\}$$

Then

$$\mathcal{S} \longleftrightarrow A_1 \times \cdots \times A_n \quad \text{with} \quad s = b_1 \cdots b_n \mapsto (b_1, \ldots, b_n)$$

Define $w(s) := \#$ 1's in s

$$\alpha_i(b_i) := \begin{cases} 0 & \text{if } b_i = 0 \\ 1 & \text{if } b_i = 1 \end{cases} \quad \Rightarrow \quad w(s) = \alpha_1(b_1) + \cdots + \alpha_n(b_n)$$

We have

$$\sum_{s \in \mathcal{S}} x^{w(s)} = \sum_{k \geq 0} \binom{n}{k} x^k \quad \left(\text{by def'n of } \binom{n}{k}\right)$$

and

$$\sum_{b_i \in A_i} x^{b_i} = 1 + x$$

Multiplication Principle

$$\Rightarrow \quad (1 + x)^n = \sum_{k \geq 0} \binom{n}{k} x^k$$

Definition. A **weight function** on S_n is called a <u>statistic</u>.

Definition. Let $w = w_1 \ldots w_n \in S_n$. A pair (w_i, w_j) is called an <u>inversion</u> if $i < j$ and $w_i > w_j$.

Remark. We want to study the function $\mathbf{inv}\colon S_n \to \mathbb{Z}_{\geq 0}$ defined by

$$\mathbf{inv}(w) := \# \text{ of inversions}$$

Definition. Let $\mathcal{I}(n,k) := \# \{w \in S_n : \mathbf{inv}(w) = k\}$

Proposition ((4.1) Generating Function for Inversions)**.**

$$\sum_{k\geq 0} \mathcal{I}(n,k)x^k = \sum_{w\in S_n} x^{\mathbf{inv}(w)} = 1 \cdot (1+x) \cdot (1+x+x^2) \cdots (1+x+\cdots+x^{n-1})$$

Example (Computing Coefficients)**.**

$$\mathcal{I}(n,1) = [x^1]\,(1 \cdot (1+x) \cdots) = (\text{choose one } x \text{ from one of the } n-1 \text{ factors}) = (n-1)$$

$$\mathcal{I}(n,2) = [x^2](1)(1+x)\cdots(1+x+\cdots+x^{n-1})$$

$$= \binom{n-1}{2} \quad \text{(choose two factors to each contribute one } x\text{)}$$

Definition. The **statistic** given by the number of cycles in a permutation is defined as follows:
For $w \in S_n$, let $C(w) := \#$ cycles in w.

Definition. Let $C(n,k)$ denote $\# \{w \in S_n : C(w) = k\}$.

Remark. We call these the **(signed) Stirling numbers of the first kind**.

Definition. The **signed Stirling numbers of the first kind** are defined as

$$C(n,k) := \# \{w \in S_n : C(w) = k\}$$

where $C\colon S_n \to \mathbb{Z}_{\geq 0}$ is the statistic given by

$$C(w) := \# \text{ cycles in } w$$

Proposition ((4.2) Basic Values)**.**

- $C(n,n) = 1$
- $C(n,1) = (n-1)!$

Theorem ((4.3) Exponential Generating Function)**.**

$$\sum_{k\geq 0} C(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1)$$

Proposition. Given any permutation, say $w = 4\,2\,7\,1\,6\,3\,5 \in S_7$, we can express it as a product of disjoint cycles. For example,

$$w = (1\ 4\ 6\ 3)(2)(5\ 7)$$

Each element i is either

1. inserted as a new cycle, or
2. inserted into an existing cycle.

This process gives rise to a combinatorial encoding: each $w \in S_n$ corresponds to a tuple $(b_1, \ldots, b_n)$ with b_i indicating how i was inserted.

Let

$$B_i := \begin{cases} \{0\} & \text{if } i \text{ starts a new cycle} \\ \{1, 2, \ldots, i-1\} & \text{otherwise} \end{cases} \quad \Rightarrow \quad b_i \in B_i$$

We define a map $S_n \longrightarrow B_1 \times \cdots \times B_n$, where

$$C(w) = \#\{i : b_i = 0\}$$

Then

$$\sum_{b_i \in B_i} x^{\beta(b_i)} = x + (i-1)x^0 = x(x+1)\cdots(x+n-1)$$

Therefore,

$$\sum_{k\geq 0} C(n,k)x^k = x(x+1)\cdots(x+n-1)$$

Proposition ((4.4) Recurrence for Stirling Numbers of the First Kind)**.**

$$C(n,k) = C(n-1,k-1) + (n-1)\,C(n-1,k)$$

Using generating functions. Recall the generating function:

$$\sum_{k\geq 0} C(n,k)x^k = x(x+1)\cdots(x+n-1)$$

We extract the x^k coefficient:

$$C(n,k) = [x^k]\,x(x+1)\cdots(x+n-1) = [x^k]\,(x+1)\cdots(x+n-2)\cdot(x+n-1)$$

Now write:

$$= [x^k]\,(x+1)\cdots(x+n-2)\cdot\sum_j C(n-1,j)x^j$$

Then:

$$= [x^k]\left(\sum_j C(n-1,j)x^j\cdot(x+n-1)\right) = \sum_j C(n-1,j)\,[x^k](x^{j+1} + (n-1)x^j)$$

To get x^k, we need:

$$j = k-1 \Rightarrow C(n-1, k-1) \quad \text{and} \quad j = k \Rightarrow (n-1)C(n-1, k)$$

Therefore:

$$C(n,k) = C(n-1,k-1) + (n-1)\, C(n-1,k)$$

□

Combinatorial proof. We analyze how to insert the last element n into a permutation of S_{n-1}:

- Case 1: Place n as a new singleton cycle. This increases the number of cycles by 1, contributing $C(n-1, k-1)$.
- Case 2: Insert n into one of the existing k cycles. There are $(n-1)$ positions available across all cycles, contributing $(n-1)C(n-1, k)$.

Thus:

$$C(n,k) = C(n-1,k-1) + (n-1)\, C(n-1,k)$$

□

Definition. The **signed Stirling numbers of the first kind**, denoted $s(n,k)$, are defined via the identity

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)\, x^k$$

Equivalently,

$$(x)_n := x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} (-1)^{n-k} C(n,k)\, x^k$$

where $C(n,k)$ are the (unsigned) Stirling numbers of the first kind.

This identity reflects the change of basis between the polynomial basis $\{x^k\}$ and the falling factorial basis $\{(x)_k\}$.

Definition. A **partition** of an n-element set X is a collection $\Pi = \{B_1, \ldots, B_k\}$ of subsets of X such that:

1. $B_i \neq \emptyset$
2. $B_i \cap B_j = \emptyset$ for $i \neq j$
3. $X = \bigcup_{i=1}^{k} B_i$

Definition. Let $S(n,k)$ denote the number of partitions of an n-element set into k **blocks**. These are called the **Stirling numbers of the second kind**.

Proposition ((4.5) Recurrence for Stirling Numbers of the Second Kind)**.**

$$S(n,k) = S(n-1,k-1) + k\, S(n-1,k)$$

Combinatorial proof. Consider how to place the element n:

- Place n in a new singleton block: contributes $S(n-1, k-1)$

- Place n into one of the k existing blocks: contributes $k \cdot S(n-1,k)$

Thus,

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

$\square$

Theorem ((Stirling Expansion))**.**

$$x^n = \sum_{k=0}^{n} k!\,S(n,k)\binom{x}{k} = \sum_{k=0}^{n} S(n,k)(x)_k$$

where $(x)_k = k!\binom{x}{k}$ denotes the falling factorial.

Theorem ((Change of Basis via Stirling Numbers))**.**

$$\sum_{k=0}^{n} S(n,k)\,x^k = (x)_n, \quad \sum_{k=0}^{n} S(n,k)\,(x)_k = x^n$$

These identities describe the change of basis between monomials and falling factorials.

Definition ((Polynomial Vector Space))**.** Let V be the **vector space of polynomials** of degree $\leq d$ over $\mathbb{R}$:

$$V = \left\{ \sum_{k=0}^{d} a_k x^k : a_0, \ldots, a_d \in \mathbb{R} \right\}$$

Two common bases for V are:

$$B_1 = \{1, x, x^2, \ldots, x^d\}$$
$$B_2 = \{1, (x)_1, (x)_2, \ldots, (x)_d\}$$

Theorem ((Power Sum via Stirling Numbers))**.**

$$1^n + 2^n + \cdots + k^n = \sum_{j=1}^{k} S(k,j) \cdot j! \cdot \binom{n+1}{j+1}$$

Definition ((Partition of an Integer))**.** A **partition** of n is a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0)$ such that

$$\sum_{i=1}^{\infty} \lambda_i = n \quad \text{(only finitely many terms are nonzero).}$$

We often write $\lambda = (\lambda_1, \ldots, \lambda_m)$ if $\lambda_{m+1} = \lambda_{m+2} = \cdots = 0$.

Notation 7. If λ is a partition of n, we write:

$$|\lambda| = n \quad \text{or} \quad \lambda \vdash n$$

The nonzero λ_i's are called the **parts** of the partition.

Definition ((Length of a Partition))**.** If $k = \#\{i : \lambda_i \neq 0\}$, then k is called the **number of parts** of λ, denoted $\ell(\lambda) = k$.

Remark. We can write partitions using exponential notation: If λ has m_1 parts equal to 1, m_2 parts equal to 2, etc., then:

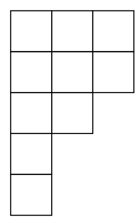
$$\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$$

For example:

$$(3, 1, 1) = 1^2 3^1$$

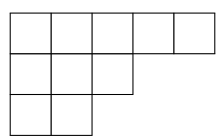
Definition ((Young Diagram / Ferrers Shape))**.** Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, the **Young diagram** (or Ferrers shape) of λ is a left-justified array of boxes with λ_i boxes in row i.

Example: For $\lambda = (3, 3, 2, 1, 1)$, the diagram is:



Definition ((Conjugate Partition))**.** The **conjugate partition** λ' is the one corresponding to the transpose of the Young diagram of λ.

Example: The conjugate of $(3, 3, 2, 1, 1)$ is $\lambda' = (5, 3, 2)$:



Lemma. The number of partitions of n with **largest part** $\leq k$ is equal to the number of partitions of n with $\leq k$ parts.

Proof. Taking the conjugate of a Young diagram reflects its rows and columns. So:

$$\#\text{ parts} = \text{length of first column}, \quad \text{largest part} = \text{length of first row}$$

Conjugation defines a bijection between the two sets. $\square$

Theorem ((Euler))**.**

$$\sum_{n \geq 0} p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

This is the generating function for the partition function $p(n)$, which counts the number of integer partitions of n.

Theorem ((Euler – Partitions into Distinct Parts))**.** Let $q(n)$ denote the number of partitions of n into **distinct parts**.

$$\sum_{n\geq 0} q(n)x^n = \prod_{i=1}^{\infty}(1+x^i)$$

This is the generating function for partitions where each part appears at most once.

Example. For $n = 5$, the valid partitions into distinct parts are:

$$(5), \quad (4,1), \quad (3,2)$$

and the invalid ones (repeated parts) are:

$$(3,1,1), \quad (2,2,1), \quad (1,1,1,1,1)$$

Remark. This generating function converges by the ratio test. Each term $1 + x^i$ corresponds to either including or excluding the part i.

Theorem ((Euler – Partitions into Odd Parts))**.** Let $p_{\text{odd}}(n)$ be the number of partitions of n where each part is **odd**. Then:

$$\sum_{n\geq 0} p_{\text{odd}}(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}} = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

Example. For $n = 5$, the partitions into odd parts are:

$$(5), \quad (3,1,1), \quad (1,1,1,1,1)$$

Remark. This generating function runs over all odd indices $2i - 1$, representing the inclusion of odd parts only.

Theorem ((Euler))**.** Let $p_{\text{odd}}(n)$ be the number of partitions of n into **odd parts**, and let $q(n)$ be the number of partitions of n into **distinct parts**. Then:

$$p_{\text{odd}}(n) = q(n)$$

Proof. We compare generating functions.

The generating function for partitions into odd parts is:

$$\sum_{n\geq 0} p_{\text{odd}}(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}$$

The generating function for partitions into distinct parts is:

$$\sum_{n\geq 0} q(n)x^n = \prod_{i=1}^{\infty}(1+x^i)$$

Now,

$$\prod_{i=1}^{\infty}(1+x^i) = \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots} = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

Hence,

$$\sum_{n\geq 0} q(n)x^n = \sum_{n\geq 0} p_{\text{odd}}(n)x^n \quad \Rightarrow \quad q(n) = p_{\text{odd}}(n)$$

□

Definition. Let $p_{\leq k}(n)$ denote the number of partitions of n with **at most** k **parts**.

Theorem. The generating function for $p_{\leq k}(n)$ is:

$$\sum_{n\geq 0} p_{\leq k}(n)x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$$

Theorem ((Euler's Pentagonal Number Theorem))**.**

$$\prod_{k=1}^{\infty}(1-x^k) = \sum_{n\in\mathbb{Z}}(-1)^n x^{\frac{n(3n-1)}{2}} = 1 + \sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{n(3n-1)}{2}} + x^{\frac{n(3n+1)}{2}}\right)$$

Corollary ((Euler's Recurrence for $p(n)$))**.**

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

where the indices are generalized pentagonal numbers $\frac{k(3k\pm 1)}{2}$, and signs alternate in pairs.

Definition (Indicator Function)**.** For $A \subseteq S$, define the **indicator function** $\chi_A : S \to \mathbb{Z}$ by

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Observation 1. Let $A, B \subseteq S$. Then:

1. $\sum_{x\in S} \chi_A(x) = |A|$
2. $\chi_A \cdot \chi_B = \chi_{A\cap B}$
3. $\chi_{A^c}(x) = 1 - \chi_A(x)$

Proof of Inclusion-Exclusion (IEP). Define

$$F(x) = (1 - \chi_A(x))(1 - \chi_B(x)) = \chi_{A^c}(x)\chi_{B^c}(x)$$

Step 1. Note that

$$\chi_{A^c \cap B^c}(x) = \chi_{(A\cup B)^c}(x) = 1 - \chi_{A\cup B}(x)$$

Step 2. Expand:

$$F(x) = 1 - \chi_A(x) - \chi_B(x) + \chi_{A\cap B}(x)$$

Conclusion 1.

$$\sum_{x\in S}(1 - \chi_{A\cup B}(x)) = |S| - |A \cup B|$$

Conclusion 2.

$$\sum_{x\in S}(1 - \chi_A(x) - \chi_B(x) + \chi_{A\cap B}(x)) = |S| - |A| - |B| + |A \cap B|$$

□

Theorem (Inclusion-Exclusion Principle (General Version))**.** Suppose $A_1, \ldots, A_n \subseteq S$. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=1}^{n}(-1)^{j+1} \sum_{J\subseteq[n],\, |J|=j} \left|\bigcap_{i\in J} A_i\right|$$

Example.

$$= |A_1| + \cdots + |A_n| - |A_1 \cap A_2| - \cdots - |A_{n-1} \cap A_n| + |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|$$

Theorem (Inclusion-Exclusion Principle (Complement Form))**.**

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=1}^{n}(-1)^{j+1} \sum_{I\subseteq[n],\, |I|=j} \left|\bigcap_{i\in I} A_i\right|$$

$$\Rightarrow \left|S - \bigcup_{i=1}^{n} A_i\right| = \sum_{j=0}^{n}(-1)^{j} \sum_{I\subseteq[n],\, |I|=j} \left|\bigcap_{i\in I} A_i\right|$$

Definition (Euler's Totient Function)**.** Let $n \in \mathbb{Z}_{\geq 0}$. Euler's totient function is the map

$$\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{Z} \quad \text{such that} \quad \varphi(n) = \#\{x : 1 \leq x \leq n \text{ and } \gcd(x, n) = 1\}$$

Example.

$$\varphi(12) = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) \cdot 12 = 12 \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} = 4$$

The numbers relatively prime to 12 in $[1, 12]$ are:

$$1,\ 5,\ 7,\ 11$$

Theorem (Euler's Product Formula)**.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n. Then

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

Definition (Derangement)**.** A **derangement** of $[n]$ is a permutation

$$w = w(1) \dots w(n) \in S_n \quad \text{such that } w(i) \neq i \text{ for all } i \in [n]$$

(i.e., a permutation with no fixed points).

Notation 8. Let $d(n)$ denote the number of derangements of $[n]$.

Theorem (Derangement Formula)**.**

$$d(n) = n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Example.

$$d(3) = 3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 3! \left(\frac{1}{3}\right) = 2$$

Proof. Let $S = \{\text{all permutations}\}$, and let

$$A_i = \{\text{permutations where } i \text{ is a fixed point}\}$$

Then $|A_i| = (n-1)!$, and for any $I \subseteq [n]$,

$$\left| \bigcap_{i \in I} A_i \right| = (n - |I|)!$$

By the inclusion-exclusion principle:

$$\left| S - \bigcup_{i=1}^{n} A_i \right| = \sum_{j=0}^{n} (-1)^j \sum_{I \subseteq [n],\, |I| = j} \left| \bigcap_{i \in I} A_i \right| = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)! = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

□

Definition (Derangement)**.** A **derangement** of $[n]$ is a permutation

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(i.e., a permutation with no fixed points).

Notation 9. Let $d(n)$ denote the number of derangements of $[n]$.

Theorem (Derangement Formula)**.**

$$d(n) = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right)$$

Example.

$$d(3) = 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 3!\left(\frac{1}{3}\right) = 2$$

Proof. Let $S = \{\text{all permutations}\}$, and define

$$A_i = \{\text{permutations where } i \text{ is a fixed point}\}$$

Then $|A_i| = (n-1)!$, and for any $I \subseteq [n]$,

$$\left|\bigcap_{i\in I} A_i\right| = (n - |I|)!$$

Applying inclusion-exclusion:

$$\left|S - \bigcup_{i=1}^{n} A_i\right| = \sum_{j=0}^{n}(-1)^j \sum_{I\subseteq[n],\,|I|=j}\left|\bigcap_{i\in I} A_i\right| = \sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)!$$

$$= n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}$$

□

Theorem (Derangement Recurrence)**.** The number of derangements satisfies the recurrence:

$$d(n) = (n-1)(d(n-1) + d(n-2))$$

Proof. Consider the position of 1 in a derangement of $[n]$. It cannot map to 1, so suppose $w(1) = k$ for some $k \neq 1$. There are $n-1$ such choices.

Now consider two cases:

- If $w(k) = 1$, then we have fixed a 2-cycle $(1\ k)$, and the rest of the permutation is a derangement of $n-2$ elements.
- If $w(k) \neq 1$, then we can remove 1 and fix k, obtaining a derangement of $n-1$ elements.

So for each of the $n-1$ values of k, we get

$$d(n) = (n-1)(d(n-1) + d(n-2))$$

□

Definition (Linear Homogeneous Recurrence)**.** A sequence $h = (h_0, h_1, h_2, \dots)$ satisfies a **linear homogeneous recurrence of degree** d if there exist constants $a_0, a_1, \dots, a_d \in \mathbb{R}$ (not all zero) such that

$$\forall n \geq d, \quad a_0 h_n + a_1 h_{n-1} + \cdots + a_d h_{n-d} = 0$$

We may assume without loss of generality that $a_d \neq 0$ and $a_0 = 1$.
To generate the sequence, we need the first d terms:

$$h_d + a_1 h_{d-1} + \cdots + a_d h_0 = 0$$

Definition. Let

$$V := \{h = (u_n) : h \text{ satisfies (*) for any initial condition}\}$$

Lemma. V is a vector space over $\mathbb{C}$.

Proof. Let $h, g \in V$ and $\alpha, \beta \in \mathbb{C}$. Then $\alpha h + \beta g \in V$, since linear combinations of solutions to a linear homogeneous recurrence are again solutions. $\square$

Lemma.

$$\dim(V) = d$$

Definition (Characteristic Polynomial)**.** The polynomial

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0$$

is called the **characteristic polynomial** of the recurrence.

Remark. Suppose the characteristic polynomial has d distinct roots $r_1, \dots, r_d \in \mathbb{C}$.

Example. Consider the recurrence

$$f_n - f_{n-1} - f_{n-2} = 0$$

Its characteristic polynomial is

$$x^2 - x - 1 = 0$$

The roots are

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Lemma. Let r_0 be a root of the characteristic polynomial. Then the sequence $h_n = r_0^n$ is in V, i.e., it is a solution to the recurrence.

Proof. For all $n \geq d$,

$$r_0^n + a_1 r_0^{n-1} + \cdots + a_d r_0^{n-d} = 0$$

because the characteristic polynomial vanishes at r_0. $\square$

Lemma. The sequences $(r_1^n), \ldots, (r_d^n)$ are linearly independent.

Corollary. These sequences form a basis of V.

Theorem. If the characteristic polynomial has d distinct roots $r_1, \ldots, r_d \in \mathbb{C}$, then every solution to the recurrence is of the form

$$h_n = c_1 r_1^n + \cdots + c_d r_d^n \quad \text{for some } c_1, \ldots, c_d \in \mathbb{C}$$

Lemma. Suppose r_1 is a root of the characteristic polynomial with multiplicity m. Then the sequences

$$(r_1^n),\ (nr_1^n),\ \ldots,\ (n^{m-1}r_1^n)$$

are all solutions to the recurrence.

Proof. Let $P(x) = x^d + a_1 x^{d-1} + \cdots + a_d$ be the characteristic polynomial. Since r_1 is a root of multiplicity m, we can write

$$P(x) = (x - r_1)^m Q(x)$$

where $Q(x)$ is a polynomial of degree $d - m$ and $Q(r_1) \neq 0$.

Then,

$$\frac{d}{dx}P(x) = m(x-r_1)^{m-1}Q(x) + (x-r_1)^m Q'(x) = (x-r_1)^{m-1}\left(mQ(x) + (x-r_1)Q'(x)\right)$$

So r_1 is a root of multiplicity $m-1$ of $P'(x)$, and this pattern continues.

Now consider

$$r_1 \frac{d}{dr}\left(r^{n-d}P(r)\right) = (n-d)r_1^{n-d}P(r_1) + r_1^n \cdot P'(r_1) = 0$$

so for $n \geq d$,

$$nr_1^n + a_1(n-1)r_1^{n-1} + \cdots + a_d(n-d)r_1^{n-d} = 0$$

hence (nr_1^n) is a solution, and similarly for higher powers of n. $\square$

Theorem. Suppose r_i is a root of multiplicity m_i. Then any solution to the recurrence is of the form

$$h_n = p_1(n)r_1^n + \cdots + p_k(n)r_k^n$$

where each $p_i(n)$ is a polynomial of degree $\leq m_i - 1$.

Example (Fibonacci via Generating Functions)**.** Consider the recurrence

$$f_n = f_{n-1} + f_{n-2}, \quad \text{for } n \geq 2, \quad f_0 = 0,\ f_1 = 1$$

Define the generating function

$$F(x) = \sum_{n \geq 0} f_n x^n$$

Note:

$$\sum_{n\geq 2} f_n x^n = \sum_{n\geq 2}(f_{n-1} + f_{n-2})x^n = x\sum_{n\geq 2} f_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} f_{n-2}x^{n-2}$$

So we get:

$$F(x) - f_0 - f_1 x = x(F(x) - f_0) + x^2 F(x)$$

Plug in $f_0 = 0,\ f_1 = 1$:

$$F(x) - x = xF(x) + x^2F(x) \Rightarrow F(x)(1 - x - x^2) = x \Rightarrow F(x) = \frac{x}{1 - x - x^2}$$

Factor the denominator:

$$F(x) = \frac{x}{(1 - r_1x)(1 - r_2x)} \quad \text{where } r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}$$

Using partial fractions:

$$F(x) = \frac{\alpha}{1 - r_1x} + \frac{\beta}{1 - r_2x}$$

Solve:

$$\alpha + \beta = 0, \quad -\alpha r_2 - \beta r_1 = 1 \Rightarrow \alpha = \frac{1}{\sqrt{5}}, \quad \beta = -\frac{1}{\sqrt{5}}$$

So:

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - r_1x} - \frac{1}{1 - r_2x} \right)$$

Extracting coefficients:

$$f_n = [x^n]F(x) = \frac{1}{\sqrt{5}} \left(r_1^n - r_2^n \right)$$

Theorem. The following are equivalent:

1. The sequence (h_n) satisfies a linear recurrence of order d:

$$h_n + a_1h_{n-1} + \cdots + a_dh_{n-d} = 0 \quad \text{for all } n \geq d, \quad a_d \neq 0$$

2. The generating function

$$H(x) = \sum_{n \geq 0} h_nx^n$$

   is a rational function:

$$H(x) = \frac{F(x)}{G(x)}$$

   where $G(x) = 1 + a_1x + \cdots + a_dx^d$, and $\deg F(x) < d$.

Observation 2. If the characteristic polynomial has roots $r_1, \ldots, r_k$ with multiplicities $m_1, \ldots, m_k$, then

$$G(x) = \prod_{i=1}^{k} (1 - r_ix)^{m_i}$$

Proof of Theorem. **(1) $\Rightarrow$ (2):**
From the recurrence,

$$\sum_{n \geq d} h_nx^n + a_1 \sum_{n \geq d} h_{n-1}x^n + \cdots + a_d \sum_{n \geq d} h_{n-d}x^n = 0$$

Shift indices appropriately:

$$\Rightarrow \left(H(x) - \sum_{n=0}^{d-1} h_n x^n \right) + a_1 x \left(H(x) - \sum_{n=0}^{d-2} h_n x^n \right) + \cdots + a_d x^d H(x) = 0$$

Group terms:

$$H(x) \cdot G(x) - F(x) = 0 \quad \Rightarrow \quad H(x) = \frac{F(x)}{G(x)}$$

where $\deg F(x) \leq d - 1$.

(2) $\Rightarrow$ (1):

Suppose $H(x) = \frac{F(x)}{G(x)}$ with $G(x) = 1 + a_1 x + \cdots + a_d x^d$. Then:

$$H(x) \cdot G(x) = F(x)$$

Now take the coefficient of x^n for $n \geq d$:

$$h_n + a_1 h_{n-1} + \cdots + a_d h_{n-d} = 0 \quad \text{for all } n \geq d$$

since $\deg F(x) \leq d - 1$ implies the higher coefficients vanish. $\square$

Corollary. Let $r_1, \ldots, r_k$ be the roots of the characteristic polynomial with multiplicities $m_1, \ldots, m_k$. Then the general term of the sequence (h_n) satisfies

$$h_n = \sum_{i=1}^{k} p_i(n) r_i^n$$

where $\deg p_i(n) < m_i$.

Proof. From the generating function approach, we have

$$H(x) = \frac{F(x)}{\prod_{i=1}^{k} (1 - r_i x)^{m_i}}$$

Using partial fractions, we write

$$H(x) = \sum_{i=1}^{k} \left(\frac{\lambda_{i1}}{1 - r_i x} + \frac{\lambda_{i2}}{(1 - r_i x)^2} + \cdots + \frac{\lambda_{im_i}}{(1 - r_i x)^{m_i}} \right)$$

By Newton's binomial theorem:

$$[x^n] \frac{1}{(1 - r_i x)^j} = \binom{n + j - 1}{j - 1} r_i^n$$

Taking coefficients of x^n, we obtain

$$h_n = \sum_{i=1}^{k} \left(\sum_{j=1}^{m_i} \lambda_{ij} \binom{n + j - 1}{j - 1} \right) r_i^n$$

Define $p_i(n) := \sum_{j=1}^{m_i} \lambda_{ij} \binom{n+j-1}{j-1}$, a polynomial of degree $< m_i$. Then:

$$h_n = \sum_{i=1}^{k} p_i(n) r_i^n$$

$\square$

1 Catalan Numbers

Definition (Catalan Numbers (Euler))**.** The Catalan number C_n is the number of triangulations of a convex polygon with $n+2$ sides.

Let P_{n+2} be a convex polygon with $n+2$ sides (e.g., P_5 is a pentagon).

A **triangulation** is a collection of diagonals that do not cross except at their endpoints and which partition P_{n+2} into triangles.

Theorem. The Catalan numbers satisfy the recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k} \quad \text{with } C_0 = 1$$

Definition (Ballot Sequence)**.** A **ballot sequence** of length $2n$ is a sequence $(a_1, \ldots, a_{2n})$ with each $a_i \in \{\pm 1\}$, such that exactly n of the a_i are $+1$, and n are -1, and the partial sums are nonnegative:

$$\sum_{i=1}^{k} a_i \geq 0 \quad \text{for all } 1 \leq k \leq 2n.$$

Remark (Ballot Interpretation)**.** This arises from the classic ballot problem:

- Two candidates, A and B, receive n votes each.
- Voter preferences are revealed one at a time.
- Encode each vote as $+1$ for A and -1 for B.
- The condition that A never trails B corresponds exactly to the partial sum condition above.

Theorem (Probability of a Random Ballot Sequence)**.** The probability that a uniformly random sequence of n $+1$'s and n -1's is a ballot sequence is

$$\frac{C_n}{\binom{2n}{n}} = \frac{1}{n+1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Definition (Dyck Path)**.** A **Dyck path** of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ consisting of steps $(1,1)$ (up-steps) and $(1,-1)$ (down-steps), such that the path never goes below the x-axis.

Example. For $n=3$, one such Dyck path is illustrated as:

```
(1,1), (1,-1), (1,-1), (1,1), (1,-1), (1,1)
```

or encoded as the sequence:

```
1, 1, -1, -1, 1, -1
```

Theorem. The number of Dyck paths of length $2n$, denoted D_n, is equal to the number of ballot sequences of length $2n$, and is given by the Catalan number:

$$D_n = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem. The n-th Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

Definition (Binary Tree)**.** A **binary tree** on n vertices is an element of $\mathcal{B}_n$, where $\mathcal{B}_n$ is defined recursively as follows:

1. $\mathcal{B}_0 = \{\varnothing\}$
2. $$\mathcal{B}_n = \left\{ \begin{array}{ccc} & v & \\ / & & \backslash \\ T_1 & & T_2 \end{array} \;:\; (T_1, T_2) \in \mathcal{B}_k \times \mathcal{B}_\ell,\ k + \ell = n \right\}$$

Here, v is called the *root*.

Theorem. The number of binary trees on $n+1$ vertices is given by the Catalan number C_n.

Proof (1). We have $|\mathcal{B}_0| = 1$, and the recursive relation:

$$|\mathcal{B}_n| = \sum_{k+\ell=n} |\mathcal{B}_k| \cdot |\mathcal{B}_\ell|$$

This recurrence defines the Catalan numbers. $\square$

Proof (2). Let $\mathcal{T}_{n+2}$ be the set of triangulations of an $(n+2)$-gon. There is a bijection between such triangulations and binary trees with $n+1$ vertices, proving that the number of such binary trees is C_n. $\square$

Definition (Plane or Catalan Tree)**.** A **plane tree** P on n vertices is an element of the set $\mathcal{P}_n$, defined recursively as follows:

1. $\mathcal{P}_1 = \{v\}$, where v is the root.
2. For $n > 1$,

$$\mathcal{P}_n = \left\{ \begin{array}{ccc} & v & \\ / & & \backslash \\ P_1 & \cdots & P_m \end{array} \;:\; P_1, \ldots, P_m \in \mathcal{P}_{k_1}, \ldots, \mathcal{P}_{k_m},\ \text{with } \sum_{i=1}^{m} k_i = n - 1 \right\}$$

That is, a plane tree consists of a root joined to an ordered sequence of subtrees $P_1, \ldots, P_m$ whose total number of vertices (excluding the root) is $n-1$. The *order* of the subtrees matters.

Theorem. The number of plane trees with $n+1$ vertices is equal to the nth Catalan number C_n.

2 Exponential Generating Functions

Definition (Exponential Generating Function)**.** Given a sequence $a_0, a_1, a_2, \ldots$, the *exponential generating function* (EGF) associated to this sequence is defined by

$$F(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n.$$

Example. Let $a_n = n!$. Then the ordinary generating function (OGF) is

$$G(x) = \sum_{n\geq 0} n! x^n,$$

whereas the exponential generating function is

$$F(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

Example. Let $D(n)$ denote the number of derangements of $[n]$. Then

$$D(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thus the exponential generating function is

$$F(x) = \sum_{n\geq 0} \left(n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^n.$$

Rewriting by interchanging the summation order:

$$F(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \sum_{n\geq k} x^n = \sum_{k\geq 0} \frac{(-1)^k}{k!} \cdot \frac{x^k}{1-x} = \frac{1}{1-x} \sum_{k\geq 0} \frac{(-x)^k}{k!} = \frac{e^{-x}}{1-x}.$$

Definition (Structure)**.** A *structure* is a function $\alpha : \{\text{finite sets}\} \to \{\text{finite sets}\}$ such that if X, Y are finite sets and $|X| = |Y|$, then

$$|\alpha(X)| = |\alpha(Y)|.$$

Definition (Exponential Generating Function of a Structure)**.** Let α be a structure. The exponential generating function (EGF) associated to α is defined by

$$F_\alpha(x) = \sum_{n\geq 0} \frac{a_n x^n}{n!},$$

where $a_n = |\alpha(X)|$ for any set X of size n.

Example (Trivial Structure)**.** Define the structure $\mathcal{E}(X) := \{*\}$ for all finite sets X. That is, $\mathcal{E}(X)$ assigns a singleton set to every input set X.

Then $|\mathcal{E}(X)| = 1$ for all X, so the exponential generating function is

$$F_{\mathcal{E}}(x) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x.$$

Example (Trivial Structure Minus the Empty Set)**.** Define $\overline{\mathcal{E}}(X)$ as

$$\overline{\mathcal{E}}(X) := \begin{cases} \varnothing & \text{if } X = \varnothing, \\ \{*\} & \text{if } X \neq \varnothing. \end{cases}$$

Then $a_n = |\overline{\mathcal{E}}(X)| = 1$ for $n \geq 1$, and $a_0 = 0$. The exponential generating function is

$$F_{\overline{\mathcal{E}}}(x) = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1.$$

Definition (Disjoint Union of Structures)**.** Given two structures α and β, define their **disjoint union structure** $\alpha \sqcup \beta$ by

$$(\alpha \sqcup \beta)(X) := \alpha(X) \sqcup \beta(X).$$

Then

$$|(\alpha \sqcup \beta)(X)| = |\alpha(X)| + |\beta(X)|.$$

Hence, $\alpha \sqcup \beta$ is a structure.

Proposition (Addition Principle for EGFs)**.** Let α and β be structures with exponential generating functions

$$F_\alpha(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}, \qquad F_\beta(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}.$$

Then the EGF of the disjoint union structure $\alpha \sqcup \beta$ is

$$F_{\alpha\sqcup\beta}(x) = F_\alpha(x) + F_\beta(x).$$

Theorem (Multiplication Principle for EGFs)**.** Let α and β be two structures, and define the product structure $\alpha \times \beta$. Then their exponential generating function satisfies

$$F_{\alpha\times\beta}(x) = F_\alpha(x) \cdot F_\beta(x),$$

where

$$F_\alpha(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}, \quad F_\beta(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}, \quad F_{\alpha\times\beta}(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}.$$

Then the coefficients satisfy

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Definition (Product structure)**.** Given two structures α and β, define their *product structure* by

$$(\alpha \times \beta)(X) := \{(\alpha(A), \beta(B)) : X = A \sqcup B\}.$$

This can be extended recursively:

$$\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n := \alpha_1 \times (\alpha_2 \times \cdots \times \alpha_n),$$

or more generally:

$$\alpha_1 \times \cdots \times \alpha_\ell := \{(\alpha_1(A_1), \ldots, \alpha_\ell(A_\ell)) : X = A_1 \sqcup \cdots \sqcup A_\ell\}.$$

Definition (Structure equivalence)**.** We say $\alpha \equiv \beta$ if they have the same cardinalities, i.e.,

$$|\alpha(X)| = |\beta(X)| \quad \text{for all finite sets } X,$$

which implies

$$F_\alpha(x) = F_\beta(x).$$

Example. Define $\alpha(X)$ to be the set of surjective functions $f : X \to [k]$. Then

$$F_\alpha(x) = \sum_{n \geq 0} S(n,k)k! \cdot \frac{x^n}{n!}$$

where $S(n,k)$ is the Stirling number of the second kind.

Define

$$\overline{E}(X) := \begin{cases} \{*\} & \text{if } X \neq \emptyset \\ \emptyset & \text{if } X = \emptyset \end{cases}$$

Then $\alpha = \overline{E} \times \cdots \times \overline{E}$ (k times), with

$$\overline{E} \times \cdots \times \overline{E} = \{(*_{A_1}, \ldots, *_{A_k}) : X = A_1 \sqcup \cdots \sqcup A_k\}$$

and

$$F_{\overline{E}}(x) = e^x - 1 \quad \Rightarrow \quad F_\alpha(x) = (e^x - 1)^k$$

Thus,

$$F_{S(\cdot,k)}(x) = \frac{1}{k!}(e^x - 1)^k$$

Example. Define $B(X)$ to be the set of unordered partitions of X. Then

$$B(X) = S(X,0) \sqcup S(X,1) \sqcup \cdots$$

Hence,

$$F_B(x) = \sum_{k \geq 0} F_{S(\cdot,k)}(x) = \sum_{k \geq 0} \frac{1}{k!}(e^x - 1)^k = e^{e^x - 1}$$

Example. Let $\mathcal{C}(X,k) = \{\omega \in S_X : \omega \text{ has } k \text{ cycles}\}$. Then the exponential generating function is

$$F_{\mathcal{C}(\cdot,k)}(x) = \sum_{n\geq 0} \frac{c(n,k)}{n!} x^n$$

where $c(n,k)$ is the number of permutations of n elements with k cycles.

- For $k=1$: $F_{\mathcal{C}(\cdot,1)}(x) = \sum_{n\geq 1} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$.
- More generally, define
  $$\mathcal{C}_o(X,k) := \{(C_1,\dots,C_k) : \omega = C_1 \cdots C_k \in S_X\}$$
  as ordered k-tuples of disjoint cycles. Then $\mathcal{C}_o \cong \mathcal{C}(\cdot,1)^k$ and
  $$F_{\mathcal{C}_o(\cdot,k)}(x) = \left(\log\left(\frac{1}{1-x}\right)\right)^k$$
  so
  $$F_{\mathcal{C}(\cdot,k)}(x) = \frac{1}{k!}\left(\log\left(\frac{1}{1-x}\right)\right)^k.$$

Definition (Partition Structure)**.** Given a structure α, we define the *partition structure* $\Pi_\alpha(X)$ as

$$\Pi_\alpha(X) := \left\{\{S_1,\dots,S_k\} : X = \bigsqcup_{i=1}^k X_i,\ S_i \in \alpha(X_i)\right\},$$

where the set $\{S_1,\dots,S_k\}$ is unordered.

Remark. This structure is related to the product structure; it collects ways to partition X and apply structure α on each part. Weak compositions are closely related.

Example. Let $\alpha = \mathcal{E}$ be the trivial structure:

$$\mathcal{E}(X) := \{*\} \quad \text{for all sets } X.$$

Then

$$\Pi_{\mathcal{E}}(X) = \left\{\{S_1,\dots,S_k\} : X = \bigsqcup_{i=1}^k X_i,\ S_i = *\right\} \cong \left\{\{X_1,\dots,X_k\} : X = \bigsqcup X_i\right\} = \mathcal{B}(X),$$

the set of unordered partitions of X.

Example. Let $\alpha = \mathcal{C}(\cdot,1)$, the structure of a single cycle on a finite set. Then

$$\Pi_\alpha(X) = \left\{\{S_1,\dots,S_k\} : X = \bigsqcup X_i,\ S_i \in \mathcal{C}(X_i,1)\right\}.$$

This corresponds to choosing a cycle permutation for each part of the partition.

Definition (Restricted Structure)**.** Given a structure α, define the restricted structure $\overline{\alpha}$ by

$$\overline{\alpha}(X) := \begin{cases} \alpha(X) & \text{if } X \neq \emptyset, \\ \emptyset & \text{if } X = \emptyset. \end{cases}$$

The exponential generating function of $\overline{\alpha}$ is related to that of α by

$$F_{\overline{\alpha}}(x) = F_{\alpha}(x) - a_0.$$

Theorem (Exponential Formula)**.** Let α be a structure and Π_α the associated partition structure. Then

$$F_{\Pi_\alpha}(x) = \exp(F_{\overline{\alpha}}(x)) = \sum_{k\geq 0} \frac{F_{\overline{\alpha}}(x)^k}{k!}.$$

Theorem (Heuristic Exponential Principle)**.** Let a_n be the number of ways to perform a certain task on an n-element set with $a_0 = 0$. Let h_n be the number of ways to partition $[n]$ into an arbitrary number of blocks and perform the task on each block. Then

$$A(x) := \sum_{n\geq 0} a_n \frac{x^n}{n!}, \qquad H(x) := \sum_{n\geq 0} h_n \frac{x^n}{n!},$$

and the relationship between the two is

$$H(x) = \exp(A(x)).$$

Example (Exponential Generating Function for Derangements)**.** The exponential generating function for the number of derangements is

$$F_D(x) = \frac{e^{-x}}{1-x}.$$

Define the structure α by

$$\alpha(X) := \begin{cases} C(X,1) & \text{if } |X| \geq 2, \\ \emptyset & \text{if } |X| = 0 \text{ or } 1, \end{cases}$$

where $C(X,1)$ denotes permutations with one cycle (i.e., cyclic permutations), and this ensures all cycles have length at least 2.

Claim: $\Pi_\alpha = D$, where D denotes the derangement structure, and

$$F_D(x) = \exp(F_{\overline{\alpha}}(x)).$$

We compute:

$$\begin{aligned} F_{\overline{\alpha}}(x) &= F_{C(\cdot,1)}(x) - x \\ &= \log\left(\frac{1}{1-x}\right) - x, \\ F_D(x) = \exp(F_{\overline{\alpha}}(x)) &= \exp\left(\log\left(\frac{1}{1-x}\right) - x\right) = \frac{e^{-x}}{1-x}. \end{aligned}$$

Proposition (OGFs vs EGFs)**.**

OGFs (Ordinary Generating Functions)	**EGFs (Exponential Generating Functions)**
Structures on *unordered* collections.	Structures on *ordered* collections.
$f(x) = \sum_{n\geq 0} a_n x^n$ $g(x) = \sum_{n\geq 0} b_n x^n$	$F(x) = \sum_{n\geq 0} \frac{a_n x^n}{n!}$ $G(x) = \sum_{n\geq 0} \frac{b_n x^n}{n!}$
Product: $\sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$	Product: $\sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \frac{x^n}{n!}$
Example: Putting n indistinguishable balls into 2 unlabeled.	**Example:** Putting n labeled balls into 2 labeled bins.
$f(x) = g(x) = \frac{1}{1-x}$ Product: $\frac{1}{(1-x)^2}$	$F(x) = G(x) = e^x$ Product: e^{2x}

Problem 2. Let h_n be the number of n-digit numbers where every digit is odd and 1 and 3 occur an even number of times.

We define the exponential generating function $H(x)$:

$$H(x) = \sum_{n\geq 0} \frac{h_n x^n}{n!}$$

Let

$$d(x) = \begin{cases} x^n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Let $F = d(x)^3$ (product over the digits 1, 3, 5, 7, 9).

$$F = \left(\frac{e^x + e^{-x}}{2}\right)^3 = \frac{1}{8}\left(e^x + e^{-x}\right)^3$$

$$= \frac{1}{8} e^x \left(e^x + 2 + e^{-x}\right) = \frac{1}{8}\left(\frac{5^n}{n!} + \frac{2\cdot 3^n}{n!} + \frac{1}{n!}\right) \Rightarrow h_n = 5^n + 2\cdot 3^n + 1$$

Problem 3. Let g_n be the number of multisets over $\{1,3,5,7,9\}$ such that the elements 1 and 3 occur an even number of times.

Let $\mathbb{M}$ be the set of multisets with elements in $\{0,1,2,\dots\}$, i.e., $\mathbb{M} \subseteq \mathbb{N}^r$. Define $\mathrm{wt}(n) = \sum n_i$.

$$g(x) = \left(1 + x^2 + x^4 + \dots\right)^2 \left(1 + x + x^2 + \dots\right)^3 = \frac{1}{(1-x^2)^2} \cdot \frac{1}{(1-x)^3}$$