Math 142

Brendan Connelly

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1 1D Discrete Time (Difference Equations)

Theorem (Difference Equation for Population Growth of One Species)**.** Let N_k denote the population at the k-th time interval (after $k\Delta t$), where N_0 is the initial population. The population growth follows the difference equation:

$$N_{k+1} = N_k + b\Delta t N_k - m\Delta t N_k$$

where:

- b is the birth rate per unit time,
- m is the mortality rate per unit time,
- Δt is the time interval.

This can be expressed as:

$$N_k = (1 + R_0\Delta t)^k N_0$$

where $R_0 = b - m$ is the net reproductive rate. Depending on the values of b and m:

- If $b > m$, $R_0 > 0$, indicating exponential growth.
- If $b < m$, $R_0 < 0$, indicating exponential decay.

Thus, N_k can be determined for any $k \geq 0$ given N_0, Δt, and R_0.

Definition (Pharmacokinetics)**.** Pharmacokinetics is the study of how drugs are absorbed, distributed, metabolized, and eliminated by the body. It involves the quantitative analysis of drug concentrations in various body compartments over time. The main parameters include absorption rates, distribution volumes, metabolism rates, and elimination half-lives, all of which help in understanding the drug's behavior within the body.

In mathematical modeling, pharmacokinetics often uses differential equations or difference equations to represent the concentration of a drug over time, taking into account factors such as dosage intervals and elimination rates.

Example (Pill Dosage and Steady-State Concentration)**.** Consider a patient taking a pill at regular intervals. Let P_k denote the drug concentration in the body after the k-th dose. The difference equation for the concentration might look like:

$$P_{k+1} = f(P_k)$$

where f is a function representing the change in drug concentration based on factors such as drug absorption and elimination.

If P_k converges as $k \to \infty$, we denote the steady-state concentration by $\hat{P}$. We find $\hat{P}$ by solving:

$$\hat{P} = f(\hat{P})$$

This fixed point represents the long-term concentration level that the drug stabilizes at if doses are administered consistently. For the specific example in the notes, the difference equation might be:

$$\hat{P} = (0.7575)\hat{P} + 40$$

where 0.7575 represents the fraction of the drug retained in the body after one interval, and 40 represents the amount of the drug added with each dose. Solving this equation yields the steady-state concentration.

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2 2D Discrete Time (Leslie Matrices)

Definition (Leslie Matrix)**.** A Leslie matrix is a square matrix used to model age-structured population dynamics, incorporating age-specific fertility and survival rates.

$$L = \begin{bmatrix} F_1 & F_2 & \cdots & F_{n-1} & F_n \\ S_1 & 0 & \cdots & 0 & 0 \\ 0 & S_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S_{n-1} & 0 \end{bmatrix}$$

Definition (Stable Age Distribution)**.** The stable age distribution is the proportion of individuals in each age class that remains constant over time when a population grows at a constant rate, assuming age-specific fertility and survival rates are fixed.

Theorem (Stable Age Distribution Using the Leslie Matrix)**.** Given a Leslie matrix L and the population vector $\vec{N}_k$ at time k, the stable age distribution $\vec{w}$ is the eigenvector corresponding to the dominant eigenvalue λ of L, where λ represents the population growth rate as $t \to \infty$. The stable age distribution is reached when:

$$\vec{N}_k = \lambda^k \vec{w}$$

as $k \to \infty$, indicating that the population distribution grows proportionally at the rate λ.

Theorem (Transition to Differential Equation)**.** Starting with the discrete model:

$$N(t + \Delta t) = N(t) + bN(t)\Delta t - dN(t)\Delta t$$

we want Δt to be small to avoid multiple births/deaths within the time interval, thus taking $\Delta t \to 0$. Rearranging the equation, we have:

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = bN(t) - dN(t)$$

Taking the limit as $\Delta t \to 0$, assuming it exists, yields the differential equation:

$$\frac{dN}{dt} = bN - dN \quad \text{or} \quad \frac{dN}{dt} = RN$$

where $R = b - d$ represents the net growth rate of the population.

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3 1D Differential Equation Models

Example (Linear Growth Model)**.** Consider the differential equation for population growth:

$$\frac{dN}{dt} = R$$

where R is the net growth rate. The solution to this differential equation is:

$$N(t) = Rt + C$$

where C is the constant of integration. If we know the initial population $N(0) = C$, then $N(t) = Rt + N_0$, where N_0 is the initial population size.

- If $R > 0$, then $N(t) \to \infty$ as $t \to \infty$.
- If $R < 0$, then $N(t) \to -\infty$ as $t \to \infty$.

Note: For $R < 0$, $N(t)$ approaching $-\infty$ is not realistic in a population model; instead, the population would approach zero.

Example (Linear Population Growth as a Function of Population)**.** If the growth rate is a linear function of the population size, we have:

$$\frac{dN}{dt} = rN(t)$$

where r is a constant. The solution is:

$$N(t) = Ce^{rt}$$

with C determined by the initial population $N(0) = C$. This describes exponential growth or decay depending on the sign of r.

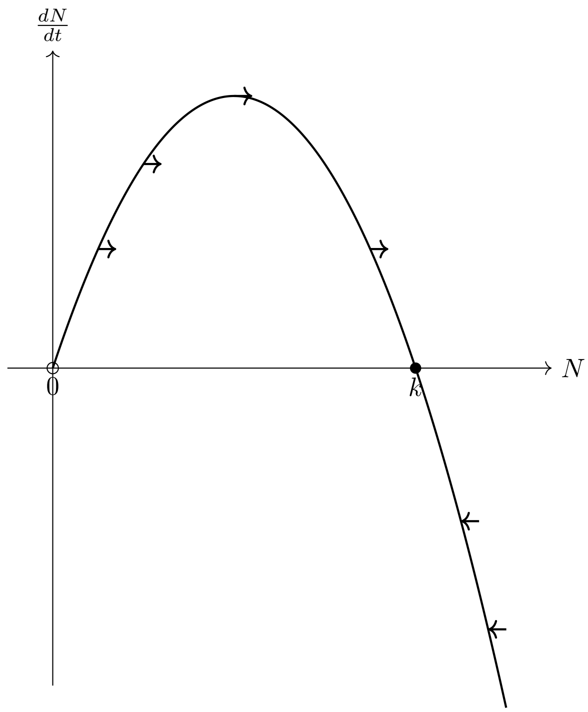
Example (Logistic Equation)**.** To incorporate a maximum population, the logistic equation is used:

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{k}\right) = rN(t) - \frac{rN(t)^2}{k}$$

where k is the *carrying capacity*, the maximum population that can be sustained by the environment. When $N(t) = k$, $\frac{dN}{dt} = 0$, indicating no population growth.

The carrying capacity represents the population size that the environment's resources can support. This model is nonlinear, making it challenging to solve analytically, so it is often analyzed graphically or numerically.

Definition (Phase Portrait)**.** A phase portrait is a graphical representation of the trajectories of a dynamical system in the phase plane, illustrating the behavior of solutions over time for different initial conditions. It shows equilibrium points, stability, and the nature of the system's behavior.



Example: The phase portrait above shows the behavior of the logistic growth model $\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$. The equilibrium points are at $N = 0$ (unstable) and $N = k$ (stable). Arrows indicate the direction of trajectories as the population N approaches or moves away from these points.

Example (Classifying Fixed Points of the Logistic Equation)**.** Consider the logistic equation:

$$\dot{N} = rN\left(1 - \frac{N}{k}\right) = f(N)$$

We have two fixed points:

$$x^* = 0 \quad \text{and} \quad x^* = k$$

To determine their stability, we find the derivative:

$$f'(N) = r - \frac{2rN}{k}$$

Evaluating at the fixed points:

- $f'(0) = r > 0$: The fixed point at $N = 0$ is **unstable**.
- $f'(k) = -r < 0$: The fixed point at $N = k$ is **stable**.

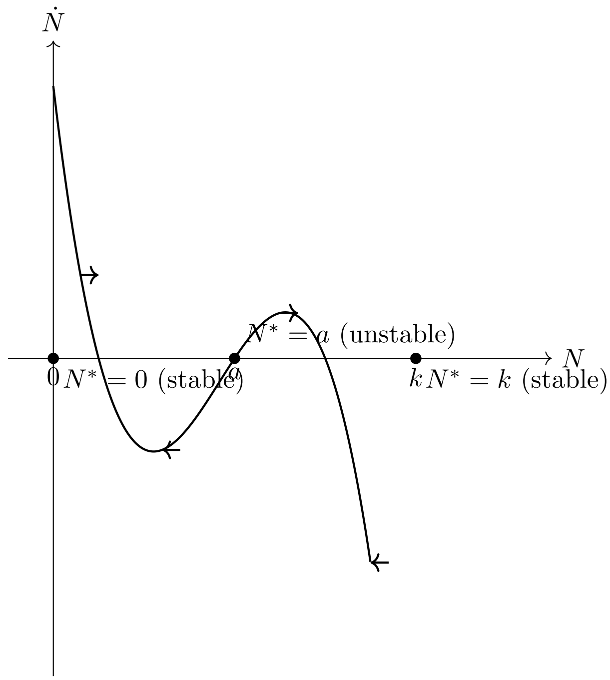
Example (Logistic Equation with the Allee Effect)**.** A small population may not be able to survive due to other influences (e.g., competitors stealing all the food). To incorporate this, we modify our logistic model as follows:

$$\dot{N} = rN\left(1 - \frac{N}{k}\right)(N - a)$$

where:

- $r > 0$ is the reproduction rate,
- $k > a > 0$ where a is the smallest sustainable population and k is the carrying capacity (largest sustainable population).

Phase Portrait:



Theorem. If $\frac{dy}{dx} = g(y)$ has a solution $y(x)$, then either:

- $y(x)$ is a constant (because $y(0) = y^*$ is a fixed point), or
- $y(x)$ is monotonic (either increasing or decreasing).

Proof. We want to show that $y(x)$ cannot change from increasing to decreasing. Suppose we have a solution $y(x)$ that goes from increasing to decreasing. Then there are two values, x_1 and x_2, such that $y(x_1) = y(x_2) = y$.

- At x_1, y is increasing, so $y'(x_1) \geq 0$.
- At x_2, y is decreasing, so $y'(x_2) \leq 0$.

Since $y(x)$ solves the differential equation $\frac{dy}{dx} = g(y)$, we have:

$$y'(x) = g(y(x)) = g(y)$$

If $y'(x_1) \geq 0$ and $y'(x_2) \leq 0$, then $g(y) = 0$. This implies that y is a fixed point, contradicting the assumption that $y(x)$ changes from increasing to decreasing unless it is constant. □

4 2D Differential Equation Models

Definition (Two-Dimensional Linear System)**.** A two-dimensional linear system is a system of the form:

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy$$

where a, b, c, d are parameters. These systems can always be written in matrix form:

$$\dot{\vec{x}} = A\vec{x} \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Theorem. These systems are linear: if $\vec{x}_1$ and $\vec{x}_2$ are solutions, then any linear combination $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution.

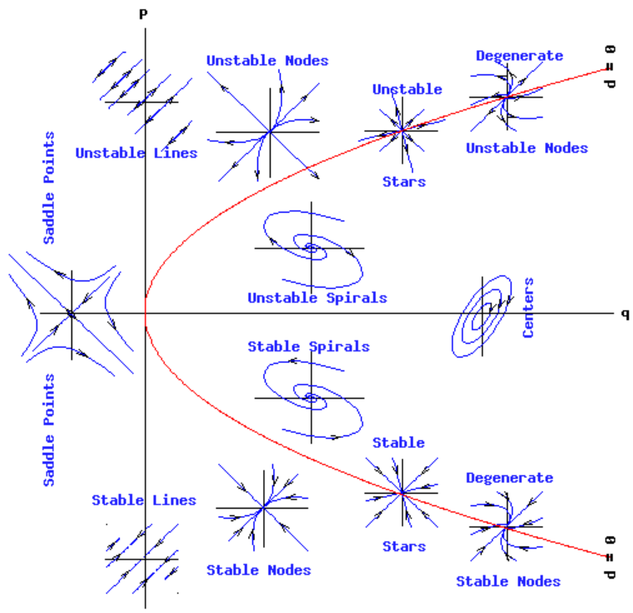
$\vec{x}^* = 0$ is always a fixed point (the origin) for any choice of A.

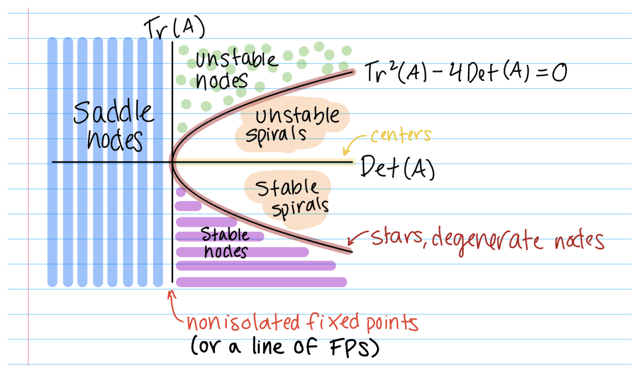
Definition (Stability)**.**

- x^* is an **attracting fixed point** if all trajectories that start near x^* approach it as $t \to \infty$, i.e., $x(t) \to x^*$.
- x^* is **globally stable** if it is attracting for trajectories starting anywhere in the plane.
- x^* is **Lyapunov stable** if all trajectories that start close to x^* remain close for all time.
- If x^* is Lyapunov stable and attracting, it is **asymptotically stable**.
- x^* is **neutrally/marginally stable** if it is attracting but not Lyapunov stable.
- x^* is **unstable** if it is not stable (i.e., it does not satisfy any of the above criteria).

Definition (Types of Equilibrium Points in Dynamical Systems)**.**

- **Stable Node**: An equilibrium point where all trajectories approach the fixed point directly over time. As $t \to \infty$, all solutions starting near the point move toward it.
- **Unstable Node**: An equilibrium point where all trajectories move away from the fixed point directly over time. As $t \to \infty$, all solutions starting near the point diverge.
- **Saddle Point**: An equilibrium point with both stable and unstable directions. Trajectories approach the fixed point along one direction (stable) but move away in another direction (unstable).
- **Stable Star**: An equilibrium point where trajectories approach the fixed point directly and at the same rate from all directions. This behavior is similar to a stable node but indicates uniform convergence speed.
- **Unstable Star**: An equilibrium point where trajectories diverge from the fixed point directly and at the same rate from all directions. This behavior is similar to an unstable node but indicates uniform divergence speed.
- **Stable Focus (Spiral)**: An equilibrium point where trajectories spiral inward as time increases. As $t \to \infty$, trajectories approach the fixed point in a spiral pattern.
- **Unstable Focus (Spiral)**: An equilibrium point where trajectories spiral outward as time increases. As $t \to \infty$, trajectories move away from the fixed point in a spiral pattern.
- **Center**: An equilibrium point where trajectories form closed orbits around the fixed point. The solutions neither approach nor diverge but continue to oscillate around the point indefinitely.





5 Post Midterm Content

Definition (Lotka-Volterra Model)**.** The Lotka-Volterra model describes the interaction between two populations: predators (sharks) and prey (fish). The system is given by the following differential equations:

$$\frac{dS}{dt} = S(-k + \lambda F), \quad \frac{dF}{dt} = F(a - bF - cS),$$

where:

- $a > 0$: Growth rate of the fish population in the absence of sharks.
- $b \geq 0$: Limitation on the fish growth rate due to overcrowding or resource depletion.
- $c > 0$: How detrimental the shark population is to the fish population.
- $k > 0$: Decline rate of the shark population in the absence of fish.
- $\lambda > 0$: How beneficial the fish population is to the shark population.

The model assumes all parameters are non-negative, and $a, c, k, \lambda > 0$.

Definition (Hyperbolicity)**.** Given the dynamical system $\dot{x} = f(x)$, where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$, an equilibrium point x^* of f is said to be *hyperbolic* if none of the eigenvalues of the Jacobian J evaluated at x^* have zero real part. Specifically:

- **Sink:** $\lambda_1, \lambda_2 < 0$
- **Source:** $\lambda_1, \lambda_2 > 0$
- **Saddle:** $\lambda_1 < 0, \lambda_2 > 0$

An equilibrium is *non-hyperbolic* if at least one eigenvalue has zero real part. Examples include:

- **Center:** $\lambda_1, \lambda_2 = i\text{Im}(\mathbb{R})$ (purely imaginary eigenvalues)
- Marginal cases with at least one zero eigenvalue.

Theorem (Hartman-Grobman Theorem (Informal Statement))**.** Let x^* be a hyperbolic equilibrium point of the dynamical system $\dot{x} = f(x)$. Then, there exists a neighborhood of x^* on which the stability type of the fixed point is faithfully captured by the linearization.

Intuitively, if x^* is a hyperbolic fixed point, then locally x^* behaves like the type of fixed point classified by its linearization.

Definition (Conservative Systems)**.** Given a dynamical system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, a function $E : \mathbb{R}^n \to \mathbb{R}$ is called a *conserved quantity* if

$$\frac{d}{dt}E(x(t)) = 0$$

and E is non-constant on every open set. If a dynamical system has a conserved quantity, it is called a *conservative system.*

Definition (Special Properties of Conservative Systems)**.** Conservative systems have several important properties:

1. A conservative system cannot have any attracting fixed points.
2. Suppose $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and there exists a conserved quantity $E(x)$. If x^* is an isolated fixed point of the system and x^* is a local minimum of $E(x)$, then all trajectories sufficiently close to x^* are closed (i.e., x^* locally behaves like a center).

Definition (SIR Model with Births and Deaths)**.** The SIR model describes the dynamics of a population divided into three compartments:

- $S(t)$: Susceptible individuals.
- $I(t)$: Infected individuals.
- $R(t)$: Recovered individuals.

Incorporating births and deaths, we make the following assumptions:

1. Birth and death rates are equal (denoted by μ), maintaining a constant population.
2. All individuals are equally capable of reproducing and dying.
3. All individuals are born susceptible to infection.

The model is governed by the system of differential equations:

$$\frac{dS}{dt} = \mu N - \beta \frac{SI}{N} - \mu S, \quad \frac{dI}{dt} = \beta \frac{SI}{N} - \gamma I - \mu I, \quad \frac{dR}{dt} = \gamma I - \mu R,$$

where:

- μ: Birth and death rate.
- β: Transmission rate of the disease.
- γ: Recovery rate.
- N: Total population size (assumed constant).

The model has two equilibria:

- **Disease-Free Equilibrium (DFE):** $(S^*, I^*, R^*) = (N, 0, 0)$.
- **Endemic Equilibrium:** $(S^*, I^*, R^*) = \left(\frac{\gamma+\mu}{\beta} N, \frac{\mu}{\beta}\left(N\frac{\beta}{\gamma+\mu} - 1\right), N - S^* - I^*\right)$.

Linear stability analysis can be used to study the behavior near these equilibria.

Definition (Simple Stochastic Mathematical Model)**.** A stochastic mathematical model describes a system in which outcomes are probabilistic rather than deterministic. For example, consider the modeling of births in a population where each individual has a probability of giving birth during a small time interval Δt.

1. The probability of an individual giving birth in time Δt is $\lambda \Delta t$.
2. The probability of not giving birth is $1 - \lambda \Delta t$.
3. For N individuals, the probability of no births is:

$$\delta_N = (1 - \lambda \Delta t)^N.$$

4. The probability of at least one birth among N individuals is:

$$\sigma_N = 1 - \delta_N = 1 - (1 - \lambda \Delta t)^N.$$

   Using the approximation $(1 - \lambda \Delta t)^N \approx 1 - \lambda N \Delta t$ for small Δt, we find:

$$\sigma_N \approx \lambda N \Delta t.$$

Combining this, the time evolution of the probability $P_N(t)$ for the population can be expressed as:

$$P_N(t + \Delta t) \approx \lambda(N-1)\Delta t P_{N-1}(t) + (1 - \lambda N \Delta t) P_N(t).$$

This model incorporates randomness and is valid under the assumption that higher-order terms of Δt (e.g., $\Delta t^2, \Delta t^3, \ldots$) are negligible.