Math 131B Running Notes

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1 Preliminaries

Definition (Metric)**.** Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if it satisfies the following properties for all $x, y, z \in X$:

(i) $d(x, x) = 0$;

(ii) $d(x, y) > 0$ if $x \neq y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, z) \leq d(x, y) + d(y, z) \quad$ (*triangle inequality*).

Definition (Metric Space)**.** The pair (X, d) is called a *metric space*, where $d(x, y) = |x - y|$ defines the metric.

Definition (Taxicab Metric)**.** The *taxicab metric* on $\mathbb{R}^n$ is defined as:

$$d_{\text{taxi}}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|.$$

This is a metric.

Definition (Discrete Metric)**.** Let X be a non-empty set. The *discrete metric* on X is defined as:

$$d_{\text{disc}}(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{otherwise.} \end{cases}$$

Definition (Euclidean Metric on $\mathbb{R}$)**.** The *Euclidean metric* on $\mathbb{R}$ is defined as:

$$d(x, y) = |x - y|,$$

where $x, y \in \mathbb{R}$.

Definition (Euclidean Metric on $\mathbb{R}^n$)**.** The *Euclidean metric* (or ℓ^2-metric) on $\mathbb{R}^n$ is defined as:

$$d_{\text{Euclidean}}(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}},$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Definition (Convergence of a Sequence in a Metric Space)**.** Let (X, d) be a metric space. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. We say that $(x_n)_{n=1}^{\infty}$ *converges* to $x_0 \in X$ (denoted $x_n \to x_0$) if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x_0) < \epsilon.$$

Remark. The condition $d(x_n, x_0) < \epsilon$ is equivalent to $|d(x_n, x_0) - 0| < \epsilon$.
Thus, $x_n \to x_0$ if and only if

$$\lim_{n \to \infty} d(x_n, x_0) = 0.$$

Proposition. Let $(x_n)_{n=1}^{\infty}$ be a sequence in some discrete metric space (X, d_{disc}).
If $(x_n)_{n=1}^{\infty}$ converges, then the sequence is eventually constant.

Proposition. Let $(x^{(k)})_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, where each $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$.
With the standard Euclidean metric, the sequence $(x^{(k)})_{k=1}^{\infty}$ converges to $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ if and only if each component sequence $(x_i^{(k)})_{k=1}^{\infty}$ converges to x_i in $\mathbb{R}$ for all $i = 1, 2, \ldots, n$.

Remark. If $(x_n)_{n=1}^{\infty}$ is eventually constant, then it converges in any metric space.

Proposition (Uniqueness of Limits)**.** Limits of sequences are unique.
Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X, d). Suppose $x_n \to x \in X$ and $x_n \to y \in X$. Then $x = y$.

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2 Point Set Topology

Definition. Let (X, d) be a metric space. Let $x_0 \in X$ and $r \in \mathbb{R}_{>0}$. We define the *ball* centered at x_0 with radius r as

$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}.$$

Definition. Let (X, d) be a metric space. Let $U \subseteq X$. We say that U is *open* if

$$\forall x \in U, \exists r > 0 \text{ such that } B(x, r) \subseteq U,$$

where $B(x, r)$ denotes the ball centered at x with radius r.

Definition. Let (X, d) be a metric space, and let $E \subseteq X$.

(i) A point $x_0 \in X$ is an *interior point* of E if $\exists r > 0$ such that $B(x_0, r) \subseteq E$.

(ii) A point $x_0 \in X$ is an *exterior point* of E if $\exists r > 0$ such that $B(x_0, r) \cap E = \emptyset$.

(iii) A point $x_0 \in X$ is a *boundary point* of E if it is neither an interior point nor an exterior point of E.

(iv) A point $x_0 \in X$ is an *adherent point* of E if $\forall r > 0$, $B(x_0, r) \cap E \neq \emptyset$.

Let $E \subseteq X$ be a subset of a metric space (X, d). We use the following **notations:**

- $\text{int}(E) := \{\text{interior points of } E\}$
- $\text{ext}(E) := \{\text{exterior points of } E\}$
- $\partial E := \{\text{boundary points of } E\}$
- $\overline{E} := \{\text{adherent points of } E\}$ (closure of E).

Definition. Let (X, d) be a metric space, and let $E \subseteq X$. We say that E is **closed** if it contains all of its adherent points, i.e.,

$$\overline{E} \subseteq E.$$

Remark. Let $E \subseteq X$, where (X, d) is a metric space. The following hold:

(i) $\text{int}(E) \subseteq E$, with equality if and only if E is open.

(ii) $E \subseteq \overline{E}$, with equality if and only if E is closed.

(iii) $\text{ext}(E) \cap E = \emptyset$, where $\text{ext}(E)$ denotes the set of exterior points of E.

(iv) E is closed if and only if $\overline{E} \subseteq E$.

Proposition. Let (X, d) be a metric space. Let $x_0 \in X$ and $R > 0$. Then, the ball $B(x_0, R)$ is open.

Fact. Let (X, d) be a metric space.

(i) $\emptyset$ is open and closed.

(ii) X is open and closed.

(iii) If $\{U_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} U_i$ is open. (Countable?)

(iv) If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

(v) If U, V are open, then $U \cup V$ is open.

- By induction, finite unions of open sets are open.

(vi) If E, F are closed, then their finite union $E \cup F$ is closed.

(vii) $\text{int}(E)$ is always open.

(viii) $\overline{E}$ is always closed.

Definition (Subspaces (1.3))**.** If (X, d) is a metric space and $Y \subseteq X$, then $(Y, d|_Y)$ is a metric space as well, obtained by restricting d to points of Y.

Definition (Relative Openness and Closedness)**.** Let (X, d) be a metric space, $Y \subseteq X$, and $E \subseteq Y$.

- E is called *relatively open* in Y if E is open in (Y, d).
- E is called *relatively closed* in Y if E is closed in (Y, d).

Definition (Subballs)**.**

$$B_X(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$$
$$B_Y(y_0, r) := \{y \in Y \mid d(y_0, y) < r\}$$

Since $Y \subseteq X$, it follows that:

$$B_Y(y_0, r) = B_X(x_0, r) \cap Y.$$

Not formal name for these balls

Example. Consider $(\mathbb{R}, d_{\text{std}})$ and $Y = [0, \infty)$. We claim $[0, 1)$ is relatively open in Y. Another way to see this is to note that $[0, 1) = B_Y(0, 1)$, and we have shown that all balls are open.

Proposition. Let (X, d) be a metric space, $Y \subseteq X$, and $E \subseteq Y$. Then:

(i) E is relatively open in Y if and only if there exists $E' \subseteq X$ open such that $E = E' \cap Y$.

(ii) E is relatively closed in Y if and only if there exists $E' \subseteq X$ closed such that $E = E' \cap Y$.

Definition (Cauchy Sequence)**.** Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is called a *Cauchy sequence* if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ such that } n, m \geq N \implies d(x_n, x_m) < \epsilon.$$

Proposition. Let (X, d) be a metric space, and $E \subseteq X$. Then E is closed if and only if $X \setminus E$ is open.

Proof. **(⇒)** Suppose E is closed. Then $\overline{E} = E$.

We want to show that if $x_0 \in X \setminus E$, then $x_0 \in \text{int}(X \setminus E)$. Let $x_0 \in X \setminus E$. Since $E = \overline{E}$, $x_0 \notin \overline{E}$. This means x_0 is not adjacent to E, so x_0 is in the exterior of E. Therefore, there exists $r_0 > 0$ such that $B(x_0, r) \cap E = \emptyset$. This implies $B(x_0, r) \subseteq X \setminus E$, so $x_0 \in \text{int}(X \setminus E)$.

(⇐) Assume $X \setminus E$ is open. Thus, $X \setminus E = \text{int}(X \setminus E)$.

We want to show that $x_0 \in \overline{E} \implies x_0 \in E$. Let $x_0 \in \overline{E}$. Then, for all $r > 0$, $B(x_0, r) \cap E \neq \emptyset$. Equivalently, for all $r > 0$, $B(x_0, r) \not\subseteq X \setminus E$. Hence, $x_0 \notin \text{int}(X \setminus E)$, which implies $x_0 \notin X \setminus E$. Therefore, $x_0 \in E$, and so $\overline{E} \subseteq E$. Thus, E is closed. □

Example (Open/Closed Discrete Metric Space)**.** Let (X, d_{disc}) be a discrete metric space, and let $E \subseteq X$. Then:

Claim: E is open.

Proof. Let $x_0 \in E$. Choose $r = 1$. Then,

$$B(x_0, 1) = \{x \in X \mid d_{\text{disc}}(x, x_0) < 1\} = \{x_0\} \subseteq E.$$

Therefore, E is open. □

$$\implies \text{Every subset of a discrete metric space is open!}$$

In particular, $X \setminus E$ is open, which implies E is closed.

$$\implies \text{Every subset of a discrete metric space is both open and closed.}$$

Definition (Subsequences)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). Suppose that $n_1, n_2, n_3, \ldots$ is an increasing sequence of integers such that $n_j \geq m$ for all j, satisfying:

$$m \leq n_1 < n_2 < n_3 < \cdots.$$

Then the sequence $(x_{n_j})_{j=1}^{\infty}$ is called a *subsequence* of the original sequence $(x_n)_{n=m}^{\infty}$.

Theorem. Let $(x_n)_{n=m}^{\infty}$ be a sequence in a metric space (X, d) that converges to some limit x_0. Then every subsequence $(x_{n_j})_{j=1}^{\infty}$ of that sequence also converges to x_0.

Definition (Limit Points)**.** Suppose that $(x_n)_{n=m}^{\infty}$ is a sequence of points in a metric space (X, d), and let $L \in X$. We say that L is a *limit point* of $(x_n)_{n=m}^{\infty}$ if and only if for every $N \geq m$ and $\epsilon > 0$, there exists an $n \geq N$ such that $d(x_n, L) \leq \epsilon$.

Definition (Cauchy Sequence)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). We say that this sequence is a *Cauchy sequence* if and only if for every $\epsilon > 0$, there exists an $N \geq m$ such that $d(x_j, x_k) < \epsilon$ for all $j, k \geq N$.

Lemma (Convergent Sequences are Cauchy Sequences)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x_n)_{n=m}^{\infty}$ is also a Cauchy sequence.

Definition (Complete Metric Space)**.** A metric space (X, d) is said to be *complete* if and only if every Cauchy sequence in (X, d) is convergent in (X, d).

Proposition (1.4.12)**.** This proposition states that

(a) Let (X, d) be a metric space, and let $(Y, d|_{Y \times Y})$ be a subspace of (X, d). If $(Y, d|_{Y \times Y})$ is complete, then Y must be closed in X.

(b) Conversely, suppose that (X, d) is a complete metric space, and Y is a closed subset of X. Then the subspace $(Y, d|_{Y \times Y})$ is also complete.

Definition (Open Cover)**.** Let (X, d) be a metric space and $E \subseteq X$. A collection $\{U_i\}_{i \in I}$ of open sets is called an *open cover* of E if

$$E \subseteq \bigcup_{i \in I} U_i.$$

Definition (Compact Set)**.** Let (X, d) be a metric space, and let $E \subseteq X$. Then, E is said to be *compact* if every open cover of E admits a finite subcover. That is, for every collection of open sets $\{U_i\}_{i \in I}$ such that

$$E \subseteq \bigcup_{i \in I} U_i,$$

there exists a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$E \subseteq \bigcup_{j=1}^{n} U_{i_j}.$$

Definition (Bounded Set)**.** Let (X, d) be a metric space, and let $E \subseteq X$. We say that E is *bounded* if there exists a point $x_0 \in X$ and a real number $R > 0$ such that

$$E \subseteq B(x_0, R),$$

where $B(x_0, R) = \{x \in X : d(x, x_0) < R\}$ is the open ball of radius R centered at x_0.

Remark. In fact, finite sets are always compact in any metric space (X, d).

Remark (Differences between Limit Points and Adherent Points)**.** The key differences between limit points and adherent points are:

1. **Includes the Point:** A limit point does not include the point itself unless it is approached by other points in the set. An adherent point always includes the point if it belongs to the set.
2. **Neighborhood Condition:** A limit point requires every neighborhood to contain another distinct point of the set. An adherent point requires every neighborhood to contain at least one point of the set, including itself.
3. **Relation to Closure:** All limit points are in the closure, but adherent points form the entire closure, including the set itself.

Proposition. Let (X, d) be a metric space and $E \subseteq X$. If E is compact, then E must be closed and bounded.

Proof. We first show that E is bounded. Suppose E is compact. Pick any $x_0 \in X$. Note that

$$E \subseteq \bigcup_{n \in \mathbb{N}} B(x_0, n),$$

where $B(x_0, n)$ denotes the open ball of radius n centered at x_0. The collection of all such balls forms an open cover of E.

By the compactness of E, there exist finitely many indices $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{j=1}^{k} B(x_0, n_j).$$

Let $N = \max\{n_1, \ldots, n_k\}$. Then $E \subseteq B(x_0, N)$, so E is bounded.

Hence, E is bounded as desired. □

Definition (Sequential Compactness)**.** Let (X, d) be a metric space and $E \subseteq X$. We say E is **sequentially compact** if every sequence $(x_n)_{n=1}^{\infty} \subseteq E$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to some $x \in E$.

Theorem (Bolzano-Weierstrass)**.** If $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $(\mathbb{R}, d_{\text{std}})$, then there exists a subsequence that converges to some real number.

Theorem (Heine-Borel)**.** Let $E \subseteq (\mathbb{R}^n, d_{\text{std}})$. If E is closed and bounded, then E is sequentially compact.

Proposition. Let (X, d) be a metric space and $E \subseteq X$. If E is compact, then E must be closed and bounded.

Proof. We first show that E is bounded. Suppose E is compact. Pick any $x_0 \in X$. Note that

$$E \subseteq \bigcup_{n \in \mathbb{N}} B(x_0, n),$$

where $B(x_0, n)$ denotes the open ball of radius n centered at x_0. The collection of all such balls forms an open cover of E.

By the compactness of E, there exist finitely many indices $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{j=1}^{k} B(x_0, n_j).$$

Let $N = \max\{n_1, \ldots, n_k\}$. Then $E \subseteq B(x_0, N)$, so E is bounded.

Hence, E is bounded as desired. Still need to show closed. □

Definition (Sequential Compactness)**.** Let (X, d) be a metric space and $E \subseteq X$. We say E is **sequentially compact** if every sequence $(x_n)_{n=1}^{\infty} \subseteq E$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to some $x \in E$.

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3 Continuity

Definition (Continuity in Metric Spaces)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is *continuous at* $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$$

Proposition. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Proposition. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. If f is continuous at $x_0 \in X$, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$ is continuous at x_0.

Theorem. Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following statements are equivalent:

(a) f is continuous.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X that converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

(c) Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.

(d) Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in X.

Corollary (Continuity Preserved by Composition)**.** Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.

(a) If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $(g \circ f)(x) := g(f(x))$, is continuous at x_0.

(b) If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.

Corollary. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Let c be a real number.

(a) If $x_0 \in X$ and f and g are continuous at x_0, then the functions

$$f + g : X \to \mathbb{R}, \quad f - g : X \to \mathbb{R}, \quad \max(f, g) : X \to \mathbb{R}, \quad \min(f, g) : X \to \mathbb{R}, \quad \text{and } cf : X \to \mathbb{R}$$

are also continuous at x_0. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbb{R}$ is also continuous at x_0.

(b) If f and g are continuous, then the functions

$$f + g : X \to \mathbb{R}, \quad f - g : X \to \mathbb{R}, \quad \max(f, g) : X \to \mathbb{R}, \quad \min(f, g) : X \to \mathbb{R}, \quad \text{and } cf : X \to \mathbb{R}$$

are also continuous on X. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbb{R}$ is also continuous on X.

Example.

- We know that $f(x) = x$ is continuous. This implies that all polynomials are continuous.
- We also know that $f(x, y) = x$ and $g(x, y) = y$ are continuous. Thus, all multivariate polynomials, such as $x^2y + 2y^3$, are continuous.

Theorem (Continuous Maps Preserve Compactness)**.** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image

$$f(K) := \{f(x) : x \in K\}$$

of K is also compact.

Proposition (Maximum Principle)**.** Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Then f is bounded. Furthermore, if X is non-empty, then f attains its maximum at some point $x_{\max} \in X$ and also attains its minimum at some point $x_{\min} \in X$.

Definition (Uniform Continuity)**.** Let $f : X \to Y$ be a map from one metric space (X, d_X) to another (Y, d_Y). We say that f is *uniformly continuous* if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$d_Y(f(x), f(x')) < \epsilon \quad \text{whenever } x, x' \in X \text{ and } d_X(x, x') < \delta.$$

Every uniformly continuous function is continuous, but not conversely. However, if the domain X is compact, then the two notions are equivalent.

Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that (X, d_X) is compact. If $f : X \to Y$ is a function, then f is continuous if and only if it is uniformly continuous.

Definition (Disconnected and Connected Sets)**.** Let (X, d) be a metric space, and let $E \subseteq X$.

- E is *disconnected* if there exist $U, V \subseteq E$, non-empty, relatively open subsets (with respect to E), such that:

$$E = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$

- E is *connected* if it is not disconnected.

Example.

- Consider $(\mathbb{R}, d_{\text{std}})$ and let $E = \{0, 1\}$. Then E is disconnected. Define $U = \{0\}$ and $V = \{1\}$. We have:

$$E = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$

- Let $F = \mathbb{R} \setminus \{0\}$. Then F is also disconnected. Define $U = (-\infty, 0)$ and $V = (0, \infty)$, which are open in $\mathbb{R}$ and thus relatively open in F. Both U and V are non-empty, and:

$$F = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$

Theorem. Let X be a non-empty subset of the real line $\mathbb{R}$. Then the following statements are equivalent:

(a) X is connected.

(b) Whenever $x, y \in X$ and $x < y$, the interval $[x, y]$ is also contained in X.

(c) X is an interval (in the sense of Definition 9.1.1).

Theorem (Continuity Preserves Connectedness)**.** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let E be any connected subset of X. Then $f(E)$ is also connected.

Corollary (Intermediate Value Theorem)**.** Let $f : X \to \mathbb{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \le y \le f(b)$ or $f(a) \ge y \ge f(b)$. Then there exists $c \in E$ such that $f(c) = y$.