Math 131A Notes

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Definition (Algebraic Number)**.** A complex number $\alpha \in \mathbb{C}$ is called an *algebraic number* if there exists a non-zero polynomial with integer coefficients

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{Z}, \quad a_n \neq 0,$$

such that $p(\alpha) = 0$. In other words, α is a root of a polynomial with integer coefficients.

The set of all algebraic numbers is denoted by $\overline{\mathbb{Q}}$ or A.

Theorem (Rational Zeros Theorem)**.** Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \geq 1$, $c_n \neq 0$, and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$. Then c divides c_0 and d divides c_n.

In other words, the only rational candidates for solutions of the polynomial equation have the form $\frac{c}{d}$, where c divides c_0 and d divides c_n.

Definition (Radicals are not in $\mathbb{Q}$)**.** *Example 3:* $\sqrt{17}$ is not a rational number.

Proof: The only possible rational solutions of the equation

$$x^2 - 17 = 0$$

are $\pm 1, \pm 17$. None of these numbers are solutions, and thus $\sqrt{17}$ is not a rational number.

Definition (Order on $\mathbb{Q}$)**.** The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying the following properties:

- O1. Given a and b, either $a \leq b$ or $b \leq a$.
- O2. If $a \leq b$ and $b \leq a$, then $a = b$.
- O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
- O4. If $a \leq b$, then $a + c \leq b + c$.
- O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Definition (Consequences of the Field Properties)**.** The following are consequences of the field properties for $a, b, c \in \mathbb{R}$:

- (i) $a + c = b + c$ implies $a = b$.
- (ii) $a \cdot 0 = 0$ for all a.

(iii) $(-a)b = -ab$ for all a, b.

(iv) $(-a)(-b) = ab$ for all a, b.

(v) $ac = bc$ and $c \neq 0$ imply $a = b$.

(vi) $ab = 0$ implies either $a = 0$ or $b = 0$.

Definition (Consequences of the Properties of an Ordered Field)**.** The following are consequences of the properties of an ordered field for $a, b, c \in \mathbb{R}$:

(i) If $a \leq b$, then $-b \leq -a$.

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv) $0 \leq a^2$ for all a.

(v) $0 < 1$.

(vi) If $0 < a$, then $0 < a^{-1}$.

(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Note: $a < b$ means $a \leq b$ and $a \neq b$.

Theorem (Triangle Inequality and Misc)**.** The following properties hold for the absolute value function for $a, b \in \mathbb{R}$:

(i) $|a| \geq 0$ for all $a \in \mathbb{R}$.

(ii) $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$.

(iii) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$ (Triangle Inequality).

Corollary (Consequence of the Triangle Inequality)**.** The following property holds for the absolute value function for $a, b \in \mathbb{R}$:

$$||a| - |b|| \leq |a - b|$$

1 Completeness

Definition (Bounded Definitions)**.** Let $\emptyset \neq A \subseteq \mathbb{R}$.

1. We say that A is *bounded above* if there exists $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$. In this case, M is called an *upper bound* for A. If moreover $M \in A$, then M is called the *maximum* of A.

2. We say that A is *bounded below* if there exists $m \in \mathbb{R}$ such that $m \leq a$ for all $a \in A$. In this case, m is called a *lower bound* for A. If moreover $m \in A$, then m is called the *minimum* of A.

3. We say that A is *bounded* if it is both bounded below and bounded above.

Definition (Supremum and Infimum)**.** Let $\emptyset eq A \subseteq \mathbb{R}$.

1. Let A be bounded above. We say L is a *least upper bound* for A if:
   (a) L is an upper bound for A.
   (b) If M is an upper bound for A, then $L \leq M$.

   This L is also called the *supremum* of A and we write $L = \sup A$.

2. Let A be bounded below. We say ℓ is a *greatest lower bound* for A if:
   (a) ℓ is a lower bound for A.
   (b) If m is a lower bound for A, then $m \leq \ell$.

   This ℓ is also called the *infimum* of A and we write $\ell = \inf A$.

Definition (Least Upper Bound and Greatest Lower Bound Properties)**.** Let $\emptyset \neq S \subseteq \mathbb{R}$.

1. We say S has the *least upper bound property* if for every nonempty subset A of S which is also bounded above, A has a least upper bound in S.
2. We say S has the *greatest lower bound property* if for every nonempty subset A of S which is also bounded below, A has a greatest lower bound in S.

Theorem (Axiom of $\mathbb{R}$)**.** The set of real numbers $\mathbb{R}$ has the least upper bound property. In fact, it is the unique ordered field with the least upper bound property. As a corollary, the set of real numbers $\mathbb{R}$ has the greatest lower bound property.

Property (Archimedean Property of $\mathbb{R}$)**.** For any $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $x < n$. This n depends on x.

Proof. Proof by contradiction. Suppose not, then there exists $x \in \mathbb{R}$ such that $x \geq n$ for all $n \in \mathbb{N}$. Hence, $\mathbb{N} \subseteq \mathbb{R}$ is bounded above. By the least upper bound property of $\mathbb{R}$, we have $\sup \mathbb{N} = L$ exists in $\mathbb{R}$. Then $L - 1$ is not an upper bound for $\mathbb{N}$, so there is an $m \in \mathbb{N}$ such that $m > L - 1$. But then $m + 1 \in \mathbb{N}$ and $m + 1 > L$, contradicting $L = \sup \mathbb{N}$.

□

Corollary (AP Corollary)**.** If $a > 0$, $b > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.

Corollary (AP Corollary)**.** For $a \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \leq a < n + 1$.

Proof. If $a \in \mathbb{Z}$, take $n = a$.

For $a > 0$ and $a \notin \mathbb{N}$, define $S = \{n \in \mathbb{Z} : n < a\} \ni 0$. We claim that there is an $m \in \mathbb{Z}$ such that $m \in S$ but $m + 1 \notin S$. If not, $m \in S$ implies $m + 1 \in S$, and we have $0 \in S$, thus by induction $\mathbb{N} \cup \{0\} \subseteq S$. This implies $\mathbb{N}$ is bounded above as S is, which is a contradiction. Take $n = m$.

For non-integer $a < 0$, we have $-a > 0$. Then there is $\ell \in \mathbb{N}$ such that $\ell < -a < \ell + 1$, and so $-\ell - 1 < a \leq -\ell$. Take $n = -\ell - 1$.

□

Corollary (AP flipped)**.** For $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$.

Definition (Density in $\mathbb{R}$)**.** Let set $A \subseteq \mathbb{R}$ be called *dense in* $\mathbb{R}$ if for any $x, y \in \mathbb{R}$ with $x < y$, there exists an $a \in A$ such that $x < a < y$.

Theorem (Rationals Dense in Reals)**.** The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. There exists $m \in \mathbb{Z}$ such that $m - 1 \leq nx < m$. Then

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

and so

$$x < \frac{m}{n} \leq x + \frac{1}{n} < y,$$

noting that $\frac{m}{n} \in \mathbb{Q}$. □

Corollary (Irrationals Dense in Reals)**.** The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then $x\sqrt{2} < y\sqrt{2}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $x\sqrt{2} < r < y\sqrt{2}$, which implies $x < \sqrt{2}r < y$. Note that $\sqrt{2}r \in \mathbb{R} \setminus \mathbb{Q}$. □

Definition (Extension to Infinity)**.** The symbols $+\infty$, $-\infty$. We adjoin these symbols with $\mathbb{R}$ so that $-\infty < a < +\infty$ for all $a \in \mathbb{R}$. If $\emptyset eq A \subseteq \mathbb{R}$ is not bounded above, we set $\sup A = +\infty$. Similarly, if $\emptyset eq A \subseteq \mathbb{R}$ is not bounded below, we set $\inf A = -\infty$.

Definition (Sequences of Real Numbers)**.** A *sequence of real numbers* is a function $f : \mathbb{N} o \mathbb{R}$. We can represent this function f as

$$f(1), f(2), \ldots$$

or $(f(n))_{n \in \mathbb{N}}$, or more commonly $(f_n)_{n \in \mathbb{N}}$, $(f_n)_{n \geq 1}$, or simply (f_n). We can also use curly braces, such as $\{f_n\}$, to denote the sequence.

Examples:

1. $(a_n)_{n \in \mathbb{N}}$ with $a_n = \frac{1}{n}$
2. $(a_n)_{n \in \mathbb{N}}$ with $a_n = (-1)^n$
3. $(a_n)_{n \in \mathbb{N}}$ with $a_n = n^2$
4. $(a_n)_{n \in \mathbb{N}}$ with $a_n = \cos\left(\frac{n\pi}{2}\right)$

2 Limits and Convergence

Definition (Convergence of a Sequence)**.** A sequence (a_n) of real numbers *converges* if there exists $a \in \mathbb{R}$ such that for any given $\epsilon > 0$, there exists an $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_\epsilon$.

In this case, a is called the *limit* of the sequence, and we write

$$a = \lim_{n\to\infty} a_n$$

or $a_n \to a$ as $n \to \infty$. We say (a_n) *converges to* a. If no such limit a exists, i.e., if the sequence does not converge, then we say the sequence *diverges*.

Theorem (Uniqueness of Limit)**.** The limit of a sequence is unique.

Proof. Assume (a_n) converges and $\lim_{no\infty} a_n = a$ and $\lim_{no\infty} a_n = b$. We want to show $a = b$.

Let $\epsilon > 0$. There exist $n_1, n_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ for all $n \geq n_1$ and $|a_n - b| < \frac{\epsilon}{2}$ for all $n \geq n_2$. Then for $n \geq \max(n_1, n_2)$, we have $|a_n - a| < \frac{\epsilon}{2}$ and $|a_n - b| < \frac{\epsilon}{2}$.

Therefore, with such n, we have

$$|a - b| \leq |a - a_n| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude $a = b$. □

Example (Limit Examples)**. Example 1** Show that (a_n) with $a_n = \frac{1}{n}$ converges to zero.

Proof. Let $\epsilon > 0$, we need to find $n_\epsilon \in \mathbb{N}$ such that $|a_n - 0| = a_n < \epsilon$ for all $n \geq n_\epsilon$. By the Archimedean property of $\mathbb{R}$, there exists $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > \frac{1}{\epsilon}$. Then for $n \geq n_\epsilon$, we have

$$\frac{1}{n} \leq \frac{1}{n_\epsilon} < \epsilon.$$

□

Example 2 Show that (a_n) with $a_n = (-1)^n$ diverges.

Proof. By contradiction. Suppose $a_n \to a \in \mathbb{R}$. Then $|a_n - a| < \frac{1}{2}$ for all $n \geq m$ for some $m \in \mathbb{N}$. For even $n \geq m$, we have $|1 - a| < \frac{1}{2}$, and for odd $n \geq m$, we have $|-1 - a| < \frac{1}{2}$. Then

$$2 = 1 + a + 1 - a \leq |1 + a| + |1 - a| < 1,$$

which is a contradiction. □

Example 3 Show that $\lim_{n\to\infty} \frac{3n+1}{5n-2} = \frac{3}{5}$.

Proof. Let $\epsilon > 0$. It is enough to show there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, we have

$$\left| \frac{3n+1}{5n-2} - \frac{3}{5} \right| < \epsilon,$$

i.e.,

$$\frac{11}{5(5n-2)} < \epsilon.$$

Note that

$$\frac{11}{5\epsilon} < 5n - 2 \iff n > \frac{2}{5} + \frac{11}{25\epsilon}.$$

So choose $n_\epsilon \in \mathbb{N}$ satisfying

$$n_\epsilon > \frac{2}{5} + \frac{11}{25\epsilon}.$$

Then for all $n \geq n_\epsilon$, we have

$$n > \frac{2}{5} + \frac{11}{25\epsilon},$$

which implies

$$\epsilon > \frac{11}{5(5n-2)} = \frac{11}{5(5n-2)}.$$

□

Theorem (Convergent Sequences are Bounded)**.** Convergent sequences are bounded.

Proof. Let (a_n) be a convergent sequence converging to $a \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq N$. Thus $|a_n| \leq |a_n - a| + |a| < 1 + |a|$ for all $n \geq N$.

Let $M = \max\{|a_1|, \ldots, |a_{N-1}|, 1 + |a|\}$, then for all $n \in \mathbb{N}$, $|a_n| \leq M$. □ □

Theorem (Limit Properties)**.** Let $(a_n), (b_n)$ be two convergent sequences with limits $a, b \in \mathbb{R}$. Then:

1. For $k \in \mathbb{R}$, we have $\lim_{n\to\infty} ka_n = ka$.
2. $\lim_{n\to\infty}(a_n + b_n) = a + b$.
3. $\lim_{n\to\infty} a_n b_n = ab$.
4. If $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a \neq 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.
5. If $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a \neq 0$, then $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.

Proof.

1. Let $\epsilon > 0$. Then there exists $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{k}$, which implies $|ka_n - ka| < \epsilon$.
2. Let $\epsilon > 0$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ and $|b_n - b| < \frac{\epsilon}{2}$. Then for $n \geq n_\epsilon = \max(n_1, n_2)$, we have $|(a_n + b_n) - (a + b)| < \epsilon$.
3. Let $\epsilon > 0$. We want to find $n_\epsilon \in \mathbb{N}$ such that $|a_n b_n - ab| < \epsilon$. Let M be such that $|a_n| \leq M$ for all n. Let n_1, n_2 be such that $|a_n - a| < \frac{\epsilon}{2(|b|+1)}$ for all $n \geq n_1$ and $|b_n - b| < \frac{\epsilon}{2M}$. Then
$$|a_n b_n - ab| \leq |a_n||b_n - b| + |b||a_n - a| < M \cdot \frac{\epsilon}{2M} + |b| \cdot \frac{\epsilon}{2(|b|+1)} < \epsilon.$$
4. Claim: $\inf\{|a_n| : n \in \mathbb{N}\} = m > 0$. Indeed, there is n_1 such that for all $n \geq n_1$, one has $|a_n - a| < \frac{|a|}{2}$, which implies $|a_n| \geq |a| - |a_n - a| \geq \frac{|a|}{2}$. So $m = \inf_n |a_n| \geq \inf\{|a_1|, \ldots, |a_{n_1}|, \frac{|a|}{2}\} > 0$.
   Now choose $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon|a|m$. Then
$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|} < \epsilon.$$
5. Combine (3) and (4).

□

Definition (Extension of Limits to Infinity)**.** For a sequence (s_n), we write $\lim s_n = +\infty$ provided that for each $M > 0$, there is a number N such that $n > N$ implies $s_n > M$.

In this case, we say the sequence *diverges to* $+\infty$.

Similarly, we write $\lim s_n = -\infty$ provided that for each $M < 0$, there is a number N such that $n > N$ implies $s_n < M$.

Example (Divergence to Infinity)**.** We need to consider an arbitrary $M > 0$ and show there exists N (which will depend on M) such that $n > N$ implies $\sqrt{n} + 7 > M$.

To see how big N must be, we "solve" for n in the inequality $\sqrt{n}+7 > M$. This inequality holds provided $\sqrt{n} > M - 7$ or $n > (M - 7)^2$. Thus, we will take $N = (M - 7)^2$.

Formal Proof.

Let $M > 0$ and let $N = (M - 7)^2$. Then $n > N$ implies $n > (M - 7)^2$, hence $\sqrt{n} > M - 7$, hence $\sqrt{n} + 7 > M$. This shows $\lim(\sqrt{n} + 7) = +\infty$.

Theorem. Let (s_n) and (t_n) be sequences such that

$$\lim_{n\to\infty} s_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} t_n > 0$$

(where $\lim_{n\to\infty} t_n$ can be finite or $+\infty$). Then

$$\lim_{n\to\infty} s_n t_n = +\infty.$$

Proof. Let $M > 0$ be given. Choose a real number m such that

$$0 < m < \lim_{n\to\infty} t_n.$$

Such an m exists because $\lim_{n\to\infty} t_n > 0$.

There are two cases to consider:

1. **Case 1** $\lim_{n\to\infty} t_n$ is finite.

   Since $\lim_{n\to\infty} t_n > m$, there exists an integer N_1 such that for all $n > N_1$,

   $$t_n > m.$$

2. **Case 2:** $\lim_{n\to\infty} t_n = +\infty$.

   In this scenario, $t_n > m$ holds for all sufficiently large n, so we can similarly find an integer N_1 such that for all $n > N_1$,

   $$t_n > m.$$

Since $\lim_{n\to\infty} s_n = +\infty$, there exists an integer N_2 such that for all $n > N_2$,

$$s_n > \frac{M}{m}.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n > N$,

$$s_n t_n > \frac{M}{m} \cdot m = M.$$

Since M was arbitrary, it follows that $\lim_{n\to\infty} s_n t_n = +\infty$. □