Math 121 Running Notes

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Metric Spaces

Definition (Metric)**.** A **metric** d on the set X is a function $d : X \times X \to \mathbb{R}$ such that:

(A) $d(x,y) \geq 0$ for all $x, y \in X$, and $d(x,y) = 0 \iff x = y$

(B) $d(x,y) = d(y,x)$ for all $x, y \in X$

(C) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$ (*triangle inequality*)

Definition. A **metric space** is a pair (X, d), a set X *equipped with a metric.*

Definition ((Subspace))**.** If (X, d) is a metric space and $Y \subseteq X$, then (Y, d) is a **subspace** metric space.

Definition ((Open Ball))**.** The **open ball** $B(x,r)$ centered at x with $r > 0$ is

$$B(x,r) := \{y \in X \mid d(x,y) < r\}$$

Definition ((Open Set))**.** A subset $Y \subseteq X$ is **open** if for all $y \in Y$, there exists $r > 0$ such that $B(y,r) \subseteq Y$
Equivalently, Y is a union of open balls

Example. Every open ball is an open set

Proposition. For every metric space (X, d), X and $\emptyset$ are open

Proposition. If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets $U_\alpha \subseteq X$, then

$$\bigcup_\alpha U_\alpha$$

is open as well
[arbitrary union of open sets is open]

Proof. Let $x \in \bigcup_\alpha U_\alpha$. Then, $\exists \alpha \in A$ such that $x \in U_\alpha$. Since U_α is open, $\exists r > 0$ such that $B(x,r) \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha$
$\Rightarrow \bigcup_\alpha U_\alpha$ is open $\square$

Proposition. If $U_1, U_2, \ldots, U_n$ are open, then $\bigcap_{i=1}^{n} U_i$ is open

Proof. Choose $r = \min\{r_1, \ldots, r_n\}$ $\square$

Definition (Closed Set)**.** A subset $Y \subseteq X$ is **closed** in X if $X \setminus Y$ is open, where

$$X \setminus Y := \{x \in X : x \notin Y\}$$

Definition (Closure)**.** The **closure** $\overline{Y}$ of Y in X is

$$\overline{Y} := \{x \in X \mid \exists \text{ sequence } (y_n)_{n=1}^{\infty} \subseteq Y \text{ with } d(y_n, x) \to 0\}$$

Proposition (Empty Set and Whole Space are Closed)**.** The sets $\emptyset$ and X are closed.

Proposition (Arbitrary Intersections of Closed Sets)**.** An arbitrary intersection of closed sets is closed.

Proposition (Finite Unions of Closed Sets)**.** A finite union of closed sets is closed.

Theorem (Convergent Sequences are Cauchy)**.** Every convergent sequence is a Cauchy sequence.

Definition (Complete Metric Space)**.** A metric space is **complete** if every Cauchy sequence converges in the space.

Example ($\mathbb{R}$ is Complete)**.** The space $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ is complete.

However, not every subspace is complete. For example, consider $(0, 1) \subset \mathbb{R}$. The sequence

$$\left(\frac{1}{n}\right)_{n=1}^{\infty} \subset (0, 1)$$

is Cauchy, but converges to $0 \notin (0, 1)$.

Thus, limits may not lie in the subspace.

Definition (Upper Bound)**.** A subset $S \subseteq \mathbb{R}$ is **bounded above** if there exists $x \in \mathbb{R}$ such that $s \leq x$ for all $s \in S$.

We call x an **upper bound**.

Theorem (Least Upper Bound Axiom)**.** If S is a non-empty subset of $\mathbb{R}$ and is bounded above, then S has a least upper bound.

That is, there exists $x = \sup(S)$ such that:

- x is an upper bound for S
- If y is also an upper bound for S, then $x \leq y$

Proposition (Closed Subset of Complete Space is Complete)**.** A closed subset of a complete metric space is complete.

Proof. Suppose $Y \subseteq X$ is a closed subset of a complete space X. Let (y_n) be a Cauchy sequence in Y. Since $Y \subseteq X$, (y_n) is also Cauchy in X. Since X is complete, there exists $x \in X$ such that $y_n \to x$.

Since Y is closed and $(y_n) \subseteq Y$, it follows that $x \in Y$. Thus, (y_n) converges in Y, and Y is complete. $\square$

Definition (Dense Subset)**.** A subset $Y \subseteq X$ is **dense** in X if $\overline{Y} = X$

$$\iff \text{Every open ball in } X \text{ contains a point of } Y$$

Theorem (Baire Category Theorem)**.** Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open dense subsets of a complete metric space X. Then

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X.

Definition (Product Spaces)**.** Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces.

Let

$$X = X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i \text{ for all } i\}$$

Proposition (Several Reasonable Metrics on X)**.**

1. **Euclidean-type metric:**

$$d(x, y) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_n(x_n, y_n)^2}$$

2. **ℓ^1-type (taxicab) metric:**

$$d(x, y) = d_1(x_1, y_1) + \cdots + d_n(x_n, y_n)$$

3. **ℓ^∞-type (maximum) metric:**

$$d(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$

Proposition. Each of the metrics defined in (1)–(3) above satisfy:

(A) A sequence $\{x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})\}_{j=1}^{\infty}$ converges to $x = (x_1, \ldots, x_n)$ if and only if $x_i^{(j)} \to x_i$ for all i.

(B) For all $x, y \in X$, we have $d_n(x_n, y_n) \leq d(x, y)$.

Theorem (Characterization of Product Topology)**.** If d is a metric on X satisfying (A), then the open sets in (X, d) are unions of open sets of the form

$$U_1 \times U_2 \times \cdots \times U_n$$

where each U_i is open in X_i.

Theorem (Open Sets in Product Metric Spaces)**.** Let d be a metric on $X = \prod_{i=1}^{n}(X_i, d_i)$ satisfying property (A). Then the open sets of X are exactly the unions of sets of the form

$$U_1 \times U_2 \times \cdots \times U_n$$

where each U_i is open in X_i.

Definition (Continuity at a Point)**.** A map $f : X \to Y$ between metric spaces is **continuous at a point** $x \in X$ if whenever $x_n \to x$, it follows that $f(x_n) \to f(x)$.

We say f is **continuous** if it is continuous at every $x \in X$.

Proposition (Easy Properties)**.**

1. The identity map $\mathrm{id} : X \to X$ is continuous.
2. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then the composition $g \circ f : X \to Z$ is continuous.

Lemma (Epsilon-Delta Characterization of Continuity)**.** A map $f : X \to Y$ is continuous at $x \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \epsilon$$

Equivalently,

$$f(B(x, \delta)) \subseteq B(f(x), \epsilon)$$

Theorem (TFAE for Continuity)**.** The following are equivalent for a map $f : X \to Y$ between metric spaces:

1. f is continuous
2. For every $x \in X$ and every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \epsilon$$

3. For every open set $U \subseteq Y$, the preimage $f^{-1}(U) \subseteq X$ is open

Definition (Epsilon-Delta Continuity)**.** A function $f : X \to Y$ between metric spaces is **continuous at** $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \epsilon$$

Definition (Projection Map)**.** Let $\pi_k : \prod_{i=1}^{n} X_i \to X_k$ denote the projection map onto the k-th coordinate.

Lemma (Continuity of Projection Maps)**.** The projection map $\pi_k : X \to X_k$ is continuous for all k.

In particular, if $U_i \subseteq X_i$ is open for each i, then

$$U_1 \times \cdots \times U_n$$

is open in $X = \prod_{i=1}^{n} X_i$.

Proof. ($\Rightarrow$) If U_i is open in X_i, then

$$\pi_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n$$

is open in X. Taking finite intersections:

$$\bigcap_{i=1}^{n} \pi_i^{-1}(U_i)$$

is open, hence $U_1 \times \cdots \times U_n$ is open.

($\Leftarrow$) Conversely, assume $U_k \subseteq X_k$ is open. Then

$$\pi_k^{-1}(U_k) = X_1 \times \cdots \times X_{k-1} \times U_k \times X_{k+1} \times \cdots \times X_n$$

is open by assumption, so π_k is continuous. □

Definition (Open Cover)**.** A collection $\{U_\alpha\}_{\alpha \in I}$ of subsets of X **covers** a set $S \subseteq X$ if

$$\bigcup_{\alpha \in I} U_\alpha \supseteq S$$

Definition (Compactness)**.** A set $S \subseteq X$ is **compact** if every open cover of S has a finite subcover.

That is, if $\{U_\alpha\}_{\alpha \in I}$ is a cover of S with each U_α open, then there exists a finite subset $J \subseteq I$ such that

$$\bigcup_{\alpha \in J} U_\alpha$$

is a cover of S.

Definition (Totally Bounded)**.** A metric space X is **totally bounded** if for every $\epsilon > 0$, there exists a finite collection of open balls of radius ϵ that cover X.

Note: X is **bounded** if there exists $r > 0$ such that $d(x, y) < r$ for all $x, y \in X$.

Theorem (TFAE for Compactness in Metric Spaces)**.** The following are equivalent for a metric space X:

1. X is compact
2. Every sequence in X has a convergent subsequence
3. X is totally bounded and complete

Definition (Separability)**.** A metric space X is **separable** if there exists a subset $S \subseteq X$ which is countable and dense.

Equivalently, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $\overline{\{x_n\}} = X$.

Example ($\mathbb{R}^n$ is Separable)**.** $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$, so $\mathbb{R}^n$ is separable.

Lemma (Subspace of Separable Space is Separable)**.** Let (X, d) be a separable metric space. Then any subspace $Y \subseteq X$ is also separable.

[Proof omitted — see textbook.]

Theorem (Totally Bounded Implies Separable)**.** If X is totally bounded, then X is separable.

Proof. Let $\epsilon_n = \frac{1}{n}$. Since X is totally bounded, for each n, there exists a finite set of points $x_{n1}, x_{n2}, \ldots, x_{nk_n}$ such that the collection of open balls

$$\{B(x_{ni}, \tfrac{1}{n})\}_{i=1}^{k_n}$$

covers X.

Let

$$\{x_{nj} \mid n \in \mathbb{N},\, j = 1, \ldots, k_n\}$$

This set is countable.

To show it's dense: given any $x \in X$ and $\epsilon > 0$, choose n such that $\frac{1}{n} < \epsilon$. Then $x \in B(x_{ni}, \frac{1}{n})$ for some x_{ni}.

Hence,

$$d(x, x_{ni}) < \tfrac{1}{n} < \epsilon$$

□

Definition (Basis of a Metric Space)**.** A collection $\mathcal{B}$ of open sets in a metric space X is called a **basis** if every open set in X is a union of sets from $\mathcal{B}$.

Equivalently, for every open set $U \subseteq X$ and every $x \in U$, there exists $V \in \mathcal{B}$ such that

$$x \in V \subseteq U$$

Definition (Second Countable)**.** A metric space X is **second countable** if there exists a basis $\mathcal{B}$ which is countable (or at most countable).

Theorem (Second Countable $\Rightarrow$ Separable)**.** If X is second countable, then X is separable. The converse also holds.

Proof. **($\Rightarrow$)** Assume X is separable. Then there exists a countable dense subset $\{x_i\}_{i\in\mathbb{N}} \subseteq X$. Define

$$\mathcal{B} = \left\{B\left(x_i, \tfrac{1}{n}\right) : i \in \mathbb{N},\, n \in \mathbb{N}\right\}$$

Then $\mathcal{B}$ is countable. We claim that $\mathcal{B}$ is a basis for X.

Let $U \subseteq X$ be open and $x \in U$. Since U is open, there exists $r > 0$ such that $B(x, r) \subseteq U$. Choose n such that $\frac{1}{n} < \frac{r}{2}$. Since $\{x_i\}$ is dense, there exists x_i such that

$$x \in B\left(x_i, \tfrac{1}{n}\right) \subseteq B(x, r) \subseteq U$$

So $\mathcal{B}$ is a basis.

($\Leftarrow$) Now assume X is second countable, so let $\mathcal{B} = \{U_1, U_2, \ldots\}$ be a countable basis. For each i, choose $x_i \in U_i$ (if U_i is nonempty). Then

$$\{x_i\}_{i\in\mathbb{N}}$$

is countable. To show it's dense: let $x \in X$ and $B(x, r)$ be any open ball. Since $\mathcal{B}$ is a basis, there exists some $U_i \in \mathcal{B}$ such that

$$x \in U_i \subseteq B(x, r)$$

So $x_i \in U_i \subseteq B(x, r) \Rightarrow x_i \to x$, proving density. □

Theorem (Second Countable $\Rightarrow$ Every Open Cover Has a Countable Subcover)**.** If X is second countable, then every open cover of X has a countable subcover.

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X, and let $\mathcal{B}$ be a countable basis. Define

$$C \subseteq \mathcal{B}$$

by: $V \in C$ if and only if there exists $\alpha \in I$ such that $V \subseteq U_\alpha$.

Then C is countable (as a subset of a countable set), and

$$\bigcup_{V\in C} V = X$$

because $\{U_\alpha\}$ is a cover and $\mathcal{B}$ is a basis.

For each $V \in C$, pick some $U_{\alpha(V)} \supseteq V$. Then the collection $\{U_{\alpha(V)}\}_{V\in C}$ is countable and covers X:

$$\bigcup_{V\in C} U_{\alpha(V)} \supseteq \bigcup_{V\in C} V = X$$

Hence, $\{U_{\alpha(V)}\}_{V\in C}$ is a countable subcover. $\square$

1 Topological Spaces

Definition (Topology)**.** A collection $\mathcal{T} = \{U_\alpha\}_{\alpha\in I}$ of subsets of a set X is a **topology** on X if:

1. $\emptyset, X \in \mathcal{T}$
2. Any union of sets in $\mathcal{T}$ is in $\mathcal{T}$
3. Any finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$

Definition (Topological Space)**.** A **topological space** is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

Each element of $\mathcal{T}$ is called an **open set** of $(X, \mathcal{T})$.

Example (Examples of Topologies)**.**

- If (X, d) is a metric space, let $\mathcal{T} = \{\text{open sets in } (X, d)\}$. Then $(X, \mathcal{T})$ is a topological space. This is called the **metric topology**.
- For any set X, the collection $\mathcal{T} = \{\emptyset, X\}$ is a topology on X. This is called the **trivial topology**.
- For any set X, the collection $\mathcal{T} = \mathcal{P}(X)$ (all subsets of X) is a topology. This is called the **discrete topology**.
- The **cofinite topology** on X: define $U \subseteq X$ to be open if $U = \emptyset$ or $X \setminus U$ is finite. That is, U is open iff $U = \emptyset$ or U is cofinite.

Definition (Metrizable)**.** A topological space $(X, \mathcal{T})$ is called **metrizable** if there exists a metric d on X such that the metric topology induced by (X, d) is equal to $\mathcal{T}$.

Definition (Convergent Sequence)**.** Let $(X, \mathcal{T})$ be a topological space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say the sequence **converges** to a point $x \in X$ if for every open set $U \in \mathcal{T}$ such that $x \in U$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$

In this case, we write $x_n \to x$ or $\lim x_n = x$, and the point x is called the **limit** of the sequence

Definition (Closed Set)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $V \subseteq X$ is **closed** if $X \setminus V \in \mathcal{T}$.

Proposition (Properties of Closed Sets)**.**

1. $\varnothing$ and X are closed.
2. A finite union of closed sets is closed.
3. An arbitrary intersection of closed sets is closed.

Definition (Induced Topology)**.** Let $f\colon X \to (Y, \mathcal{T})$. The **induced topology** (or **initial topology**) $f^{-1}\mathcal{T}$ on X is defined as

$$f^{-1}\mathcal{T} := \{f^{-1}(U) : U \in \mathcal{T}\}$$

This is the *coarsest* topology on X that makes f continuous.

Example (Subspace Topology)**.** Let $X \subseteq Y$, and consider the inclusion map $i\colon X \hookrightarrow Y$ given by $x \mapsto x$. Then,

$$i^{-1}\mathcal{T} = \{U \cap X : U \in \mathcal{T}\}$$

This is called the **subspace topology** or **relative topology** on X.

For instance, consider $[0,1] \subset \mathbb{R}$. Then $(0, \frac{1}{2})$ is open in $[0,1]$ (under the subspace topology), but not open in $\mathbb{R}$.

Definition (Neighborhoods, Interior, and Closure)**.** Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$.

(i) A set S is a **neighborhood** of a point $x \in X$ if there exists $U \in \mathcal{T}$ such that $x \in U \subseteq S$.

Remark. Some authors assume neighborhoods are themselves open.

(ii) A point $x \in S$ is called an **interior point** of S if S is a neighborhood of x.

(iii) The **interior** of S, denoted $\operatorname{int}(S)$, is the set of all interior points of S.

Remark. $\operatorname{int}(S) \in \mathcal{T}$, i.e., it is an open set.

(iv) A point $x \in X$ is an **adherent point** (or **point of closure**) of S if every neighborhood of x intersects S.

Remark. Every point of S is adherent to S.

(v) A point $x \in X$ is a **limit point** of S if $x \notin S$ and x is adherent to S.

(vi) A point $x \in X$ is a **boundary point** of S if x is adherent to both S and $X \setminus S$. The **boundary** of S, denoted ∂S, is the set of all boundary points of S.

(vii) The **closure** of S, denoted $\overline{S}$, is the set of all adherent points of S.

Remark. $\overline{S}$ is the smallest closed set containing S.

Proposition (Local Condition for Continuity)**.** A function $f\colon X \to Y$ is continuous at a point $x \in X$ if for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that

$$f(U) \subseteq V.$$

Definition (Continuous Function)**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A function $f\colon X \to Y$ is **continuous** if for every open set $V \in \mathcal{T}'$, the preimage $f^{-1}(V) \in \mathcal{T}$, i.e.,

$$V \in \mathcal{T}' \quad \Rightarrow \quad f^{-1}(V) \in \mathcal{T}$$

Proposition (Basic Properties of Continuous Maps)**.** (1) The identity map $\mathrm{id}\colon (X, \mathcal{T}) \to (X, \mathcal{T})$ is continuous.

(2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then the composition $g \circ f\colon X \to Z$ is continuous.

Definition (Homeomorphism)**.** A continuous map $f\colon X \to Y$ is a **homeomorphism** if there exists a continuous inverse $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

We say that X and Y are **homeomorphic** if there exists a homeomorphism $f\colon X \to Y$.

We write $X \cong Y$ or $X \simeq Y$. This defines an equivalence relation on the class of topological spaces.

Example. The real line $\mathbb{R}$ is homeomorphic to any open interval (a, b), and to any ray of the form (r, ∞).

For instance:

$$\mathbb{R} \cong (0,1) \cong (-1,1) \cong (0, \infty)$$

Definition (Basis for Topological Space)**.** A **basis** $\mathcal{B}$ for a topological space $(X, \mathcal{T})$ is a subcollection of $\mathcal{T}$ such that every $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

Lemma. A collection $\mathcal{B}$ is a basis for a topology on X if and only if for every $x \in X$ and every neighborhood U of x, there exists $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a basis. Let $x \in X$, and let U be a neighborhood of x. Then there exists an open set $U' \in \mathcal{T}$ such that $x \in U' \subseteq U$. Since $\mathcal{B}$ is a basis, U' can be written as a union of elements of $\mathcal{B}$:

$$U' = \bigcup_{\alpha \in I} V_\alpha, \quad V_\alpha \in \mathcal{B}$$

So $x \in V_\alpha$ for some $\alpha \in I$ and $V_\alpha \subseteq U$, as required.

($\Leftarrow$) Suppose the right-hand side holds. Let $U \in \mathcal{T}$. Then for every $x \in U$, there exists $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U$. Thus,

$$U = \bigcup_{x \in U} V_x$$

is a union of basis elements, so $U \in \mathcal{T}$. ∎

Theorem (Criterion for a Basis)**.** A collection $\mathcal{B}$ of subsets of X is a basis for some topology on X if and only if:

1. For every $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$.
2. If $U, V \in \mathcal{B}$ and $x \in U \cap V$, then there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.

Definition (Product Topology)**.** Let $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces. The **product topology** on $X = X_1 \times \cdots \times X_n$ is the topology generated by the basis

$$\mathcal{B} = \{U_1 \times \cdots \times U_n \,|\, U_i \in \mathcal{T}_i \text{ for all } i\}$$

Remark (Projection Maps)**.** Recall that the **projection** map $\pi_i \colon X \to X_i$ is defined by

$$\pi_i(x_1, \ldots, x_n) = x_i$$

Lemma. The product topology on X is the **smallest topology** on X such that each projection map π_i is continuous.

Theorem (Lindelöf's Thm)**.** If $(X, \mathcal{T})$ is **second countable**, then every open cover has a **countable subcover**.

Proof. (Same as before) $\square$

Theorem ((Second Countable $\Rightarrow$ Separable))**.**

Proof. Let $\mathcal{B}$ be a **countable basis** for $(X, \mathcal{T})$, say $\mathcal{B} = \{U_1, U_2, \ldots\}$. Choose $x_i \in U_i$ for each i. Then $\{x_i\}_{i=1}^{\infty}$ is countable. If $U \in \mathcal{T}$ is nonempty, then $U = \bigcup_{i \in I} U_i$ for some index set I, and there exists $i \in I$ such that $U_i \subseteq U$ and $x_i \in U_i$, so $x_i \in U$. Hence, $\{x_i\}_{i=1}^{\infty}$ is **dense**. $\square$

Remark. Exercise 2.4.1 shows the converse is false.

Lemma (Open Map Proj.)**.** Each $\pi_i : X \to X_i$ is an **open map**, meaning it takes open sets to open sets.

Proof. Take $V \subset X$ open. Then

$$V = \bigcup_{\alpha \in I} U_{1\alpha} \times \cdots \times U_{n\alpha}$$

for open $U_{i\alpha} \subset X_i$. Then

$$\pi_i(V) = \bigcup_{\alpha \in I} U_{i\alpha}$$

is open since unions of open sets are open. $\square$

Lemma ((Product Map Continuity))**.** Let $X = \prod_i X_i$ be a topological space. Then $f : E \to X$ is continuous

$$\iff \pi_i \circ f : E \to X_i \text{ is continuous for all } i.$$

Definition (Quotient Topology)**.** The **quotient topology** on $X/\sim$ is given by $\mathcal{T}_\sim$, where

$$U \in \mathcal{T}_\sim \iff \pi^{-1}(U) \in \mathcal{T}$$

(Exercise: check $\mathcal{T}_\sim$ is a topology)

Lemma ((Universal Property of Quotient Topology))**.** $\mathcal{T}_\sim$ is the **largest topology** on $X/\sim$ such that π is continuous.

Proof. Suppose $\pi : (X, \mathcal{T}) \to (X/\sim, \mathcal{T}')$ is continuous. Let $U \in \mathcal{T}'$. Then $\pi^{-1}(U) \in \mathcal{T}$, so $U \in \mathcal{T}_\sim$. Hence, $\mathcal{T}' \subseteq \mathcal{T}_\sim$. ☐

Example. • $[0,1]/\sim$, where $0 \sim 1$ This identifies the endpoints of the interval to form a circle. $\pi : [0,1] \to [0,1]/\sim$, and the quotient space is homeomorphic to the circle S^1.

Example. • $[0,1]^2/\sim$, where

$$(x, 0) \sim (x, 1), \quad (0, y) \sim (1, y)$$

This glues opposite edges of the square to form a torus.

Lemma ((Descent of Continuous Map))**.** Suppose $f : X \to Y$ is continuous. If f is **constant on each equivalence class**, then there exists $\overline{f} : X/\sim \to Y$ such that

$$\overline{f}([x]) = f(x)$$

and $\overline{f}$ is well-defined.

Definition (Separation Axioms)**.** A topological space $(X, \mathcal{T})$ is called:

(1) T_1: if for all $x \neq y \in X$, there exists open $U \ni y$ such that $x \notin U$.

(2) T_2 (Hausdorff): if for all $x \neq y \in X$, there exist open $U \ni x$, $V \ni y$ such that $U \cap V = \emptyset$. ($\Rightarrow$ have **unique limits**)

(3) **Regular**: if for any closed set $E \subset X$ and $x \notin E$, there exists open $V \supset E$ and open $U \ni x$ such that $U \cap V = \emptyset$

(4) T_3: if X is **regular** and T_1

(5) **Normal**: if for each pair of disjoint closed sets $E, F \subset X$, there exist open sets $U \supset E$, $V \supset F$ such that $U \cap V = \emptyset$

Lemma. If X is normal, then X is T_1 *i.e.,* X is $\mathsf{T}_1 \iff$ singleton sets are closed.

Proof. **($\Rightarrow$)** Assume $\{x\}$ is closed for each $x \in X$. Let $y \neq x$. Then $\{x\} \subset X$ is closed, and since $y \notin \{x\}$, there exists open $U \ni y$ with $x \notin U$. $\Rightarrow \mathsf{T}_1$.

($\Leftarrow$) Assume T_1. Let $x \in X$, and show $\{x\}$ is closed. Let $y \neq x$. By T_1, there exists open U with $x \in U$, $y \notin U$. So $y \in X \setminus \{x\}$ open $\Rightarrow \{x\}$ is closed. ☐

Remark. (i) $\mathsf{T}_4 \Rightarrow \mathsf{T}_3 \Rightarrow \mathsf{T}_2 \Rightarrow \mathsf{T}_1$

(ii) T_2 (Hausdorff) $\Rightarrow$ limits of convergent sequences are unique. *(Exercise: prove this)*

Recall: $x_n \to x$ if for all open $U \ni x$, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in U$

(iii) The **cofinite topology** on an infinite set X is an example of a T_1 space which is **not** T_2

Example. The cofinite topology is T_1 but not T_2. Indeed, suppose U, V are open with $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Then $X = X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$, which is a union of two finite sets — contradiction since X is infinite. Hence, cannot separate x, y with disjoint neighborhoods.

Theorem ((Every Metric Space is T_4))**.** Every metric space is normal, hence T_4. In particular, it satisfies T_1.

Lemma. A topological space X is **normal** if and only if: For every closed set $E \subset X$ and every open set $W \supset E$, there exists an open set U such that

$$E \subset U \subset \overline{U} \subset W.$$

Theorem ((Urysohn's Lemma))**.** Suppose X is normal and $E, F \subset X$ are disjoint, closed subsets. Then there exists a continuous function $f : X \to [0, 1]$ such that

$$f|_E = 0 \quad \text{and} \quad f|_F = 1.$$

Theorem ((Tietze Extension Theorem))**.** Let X be a normal topological space, and let $Y \subset X$ be closed. Suppose $f : Y \to \mathbb{R}$ is bounded and continuous. Then there exists a continuous extension $h : X \to \mathbb{R}$ such that

$$h|_Y = f.$$

Definition. A topological space X is **compact** if every open cover of X has a finite subcover.

Definition. A subset $S \subset X$ is **compact** if S is compact with respect to the subspace topology; that is, for every open cover of S by open sets in X, there exists a finite subcover.

Proposition ((Properties of Compactness))**.** Let X be a topological space.

1. A finite union of compact subsets of X is compact.
2. If X is compact and $Y \subset X$ is closed, then Y is compact.
3. If X is Hausdorff and $Y \subset X$ is compact, then Y is closed.
4. If X is compact and Hausdorff, then X is T_4.
5. If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

Theorem ((Fundamental Theorem of Point-Set Topology))**.** Let $f : X \to Y$ be continuous, where X is compact and Y is Hausdorff. If f is injective, then $f : X \to f(X)$ is a homeomorphism.

Remark. This is very useful, since you **don't need to show the inverse is continuous or compute it at all**.

Corollary (Quotient Map from Compact to Hausdorff)**.** Let $f : X \to Y$ be continuous and surjective. Suppose X is compact and Y is Hausdorff.

Define an equivalence relation $\sim$ on X by

$$x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

Then the induced map

$$\overline{f} : X/\sim \to Y$$

is a homeomorphism.

Definition (Locally Compact)**.** A topological space X is **locally compact** if for every $x \in X$, there exists an open set $W \ni x$ such that $\overline{W}$ is compact.

Example. If X is compact, then X is locally compact.

(Just take $W = X$)

Example. The space $\mathbb{R}^n$ is locally compact but not compact.

Proposition (Closed Subsets of Compact Hausdorff Spaces are Locally Compact)**.** Let Y be a compact Hausdorff space, and let $x \in Y$. Then the subspace $X = Y \setminus \{x\}$ is locally compact.

Definition (One-Point Compactification)**.** Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. A **one-point compactification** of X is a compact Hausdorff space $(Y, \mathcal{S})$ such that:

1. $Y = X \cup \{\infty\}$ for some point $\infty \notin X$
2. $\mathcal{T} = \{U \cap X : U \in \mathcal{S}\}$, i.e., $\mathcal{T}$ is the subspace topology inherited from Y

Theorem (Existence and Uniqueness of One-Point Compactification)**.** Every locally compact Hausdorff space X admits a one-point compactification. Moreover, this compactification is unique up to homeomorphism.

Connectedness

Definition (Disjoint Union of Topological Spaces)**.** Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. The **disjoint union** of X_1 and X_2 is the space

$$X = X_1 \sqcup X_2$$

with topology

$$\mathcal{T} = \{U \subseteq X : U \cap X_i \in \mathcal{T}_i \text{ for } i = 1, 2\}.$$

This is also known as the **external direct sum** of topological spaces.

Example: $[0,1) \sqcup [0,1]$ is two copies of the interval.

Proposition. The topology $\mathcal{T}$ defined above is the **largest topology** on $X_1 \sqcup X_2$ for which the inclusion maps

$$i_1 : X_1 \hookrightarrow X, \quad i_2 : X_2 \hookrightarrow X$$

are continuous.

Definition (Disconnected Space)**.** A topological space X is **disconnected** (or **not connected**) if there exist nonempty disjoint open sets $X_1, X_2 \in \mathcal{T}$ such that

$$X = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \emptyset.$$

Equivalently, X is disconnected if it can be written as the disjoint union of two nonempty open sets.

Definition (Connected Subspace)**.** A topological space X is **connected** if it is not disconnected.

A subspace $Y \subseteq X$ is **connected** if it is connected with respect to the subspace topology inherited from X, i.e., the relative topology induced by the inclusion map $i : Y \hookrightarrow X$.

Proposition (Image of Connected Space is Connected)**.** Let $f : X \to Y$ be continuous. If X is connected, then $f(X) \subseteq Y$ is connected.

Proof. Suppose $f(X)$ is disconnected. Then there exist nonempty open sets $Y_1, Y_2 \subseteq Y$ such that

$$f(X) \subseteq Y_1 \cup Y_2, \quad Y_1 \cap Y_2 = \emptyset.$$

Since f is continuous, the preimages $f^{-1}(Y_1), f^{-1}(Y_2) \subseteq X$ are open. These preimages are disjoint, and

$$f^{-1}(Y_1) \cup f^{-1}(Y_2) = X,$$

so X is disconnected — a contradiction. □

Proposition (Connected Union with Intersections)**.** Let $\{E_\alpha\}_{\alpha \in I}$ be a collection of connected subspaces of X such that

$$E_\alpha \cap E_\beta \neq \emptyset \quad \text{for all } \alpha, \beta.$$

Then

$$E = \bigcup_{\alpha \in I} E_\alpha$$

is connected.

Definition (Connected Component)**.** Given $x \in X$, the **connected component** of x is the union of all connected subsets of X that contain x.

Equivalently, it is the **maximal connected subset** of X containing x.

Remark. This definition implicitly uses the earlier proposition: the union of connected sets with pairwise nonempty intersection is connected. Hence, the connected component of x is connected.

Proposition (Connected Components are Disjoint or Equal)**.** Let $X_1, X_2 \subseteq X$ be connected components. Then either

$$X_1 = X_2 \quad \text{or} \quad X_1 \cap X_2 = \emptyset.$$

Lemma (Path-Connectedness is an Equivalence Relation)**.** The relation "there exists a path in X from x to y" defines an equivalence relation on X.

Theorem (Path-Connected Implies Connected)**.** Every path-connected topological space is connected.

Proof. Fix $x_0 \in X$. For each $x \in X$, let $\gamma_x : [0,1] \to X$ be a path from x_0 to x. By Theorems 8.1 and 8.4, the image $\gamma_x([0,1]) \subseteq X$ is connected.

Each such path contains x_0, and the union

$$X = \bigcup_{x \in X} \gamma_x([0,1])$$

is a union of connected subsets, all containing the common point x_0. Hence, X is connected. □

Corollary (Connected Components are Unions of Path Components)**.** Let X be a topological space. Then each connected component of X is a union of path components of X.

Definition (Path in a Topological Space)**.** Let X be a topological space. A **path in** X from x_0 to x_1 is a continuous map

$$\gamma : [0,1] \to X \quad \text{such that } \gamma(0) = x_0 \text{ and } \gamma(1) = x_1.$$

Definition (Path-Connected Space)**.** A space X is **path-connected** if for every $x_0, x_1 \in X$, there exists a path $\gamma : [0,1] \to X$ from x_0 to x_1.

Theorem (Path-Connected Implies Connected)**.** If X is path-connected, then X is connected.

Proof. Suppose X is path-connected but not connected. Then there exist nonempty disjoint open sets $X_1, X_2 \subseteq X$ such that

$$X = X_1 \cup X_2, \quad X_1 \cap X_2 = \emptyset.$$

Pick $x_1 \in X_1$, $x_2 \in X_2$. Since X is path-connected, there exists a continuous path

$$\gamma : [0,1] \to X \quad \text{with } \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Then $\gamma^{-1}(X_1)$ and $\gamma^{-1}(X_2)$ are nonempty, disjoint, open subsets of $[0,1]$, and

$$\gamma^{-1}(X_1) \cup \gamma^{-1}(X_2) = [0,1].$$

This expresses $[0,1]$ as a disjoint union of two nonempty open sets — contradicting the fact that $[0,1]$ is connected.

Thus, X must be connected. □

Definition (Path Equivalence Relation)**.** Let X be a topological space. Define a relation $x \sim y$ if there exists a path in X from x to y.

Then $\sim$ is an equivalence relation:

1. **Reflexivity:** $x \sim x$ via the constant path $\gamma(t) = x$.
2. **Symmetry:** If $\gamma : [0,1] \to X$ is a path from x to y, then the reversed path
$$\widetilde{\gamma}(t) := \gamma(1-t)$$
is a path from y to x.
3. **Transitivity:** If γ_1 is a path from x to y and γ_2 is a path from y to z, define the concatenated path $\gamma_3 : [0,1] \to X$ by
$$\gamma_3(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$
Then γ_3 is continuous and a path from x to z.

We call $\gamma_3 = \gamma_1 \cdot \gamma_2$ the **concatenation** of γ_1 and γ_2.

Remark. To rigorously show continuity of γ_3, observe:

- $[0, \frac{1}{2}] \xrightarrow{\gamma_1(2t)} X$ and $[\frac{1}{2}, 1] \xrightarrow{\gamma_2(2t-1)} X$ are continuous.
- Both pieces agree at $t = \frac{1}{2}$, so we obtain a well-defined map on the glued interval.
- By the **First Topology Pasting Lemma**, γ_3 is continuous.

Definition (Path Component)**.** The equivalence classes of X under the path-connectedness relation $\sim$ are called the **path components** of X.

Proposition (Product of Connected Spaces is Connected)**.** Let X_i be connected for all $i = 1, \ldots, n$. Then the product
$$X_1 \times X_2 \times \cdots \times X_n$$
is connected.

Proposition (Product of Path-Connected Spaces is Path-Connected)**.** If each X_i is path-connected for $i = 1, \ldots, n$, then the product
$$X_1 \times X_2 \times \cdots \times X_n$$
is path-connected.

Proposition (Quotient of Connected Space is Connected)**.** Let X be connected, and let $\sim$ be any equivalence relation on X. Then the quotient space $X/\sim$ is connected.

Proposition (Quotient of Path-Connected Space is Path-Connected)**.** Let X be path-connected, and let $\sim$ be any equivalence relation on X. Then the quotient space $X/\sim$ is path-connected.

Product Topology for Infinite Product Spaces

Definition (Infinite Product Space)**.** Let A be an indexing set. For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space with $X_\alpha \neq \emptyset$.

Define the product space as

$$X := \prod_{\alpha \in A} X_\alpha = \left\{ x : A \to \bigsqcup_{\alpha \in A} X_\alpha \;\middle|\; x(\alpha) \in X_\alpha \text{ for all } \alpha \in A \right\}$$

This is the set of all functions choosing one point from each X_α.

Remark (Axiom of Choice)**.** To guarantee that $X \neq \emptyset$, we need the **Axiom of Choice**: Given a family $\{S_\alpha\}_{\alpha \in A}$ of nonempty sets, there exists a function $f : A \to \bigsqcup_{\alpha \in A} S_\alpha$ such that $f(\alpha) \in S_\alpha$ for all $\alpha \in A$.

Definition (Projection Map)**.** For each $\alpha \in A$, define the **projection map**

$$\pi_\alpha : X \to X_\alpha, \quad x \mapsto x(\alpha).$$

Definition (Product Topology)**.** The **product topology** on $X = \prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis

$$\left\{ \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \;\middle|\; \{\alpha_1, \ldots, \alpha_n\} \subset A \text{ finite, } U_i \subseteq X_{\alpha_i} \text{ open} \right\}.$$

Remark. If A is finite, then the infinite product topology coincides with the product topology defined previously for finite products.

Zorn's Lemma

Definition (Partially Ordered Set (Poset))**.** A **partially ordered set** (or **poset**) S is a nonempty set equipped with a binary relation $\leq$ such that:

1. $x \leq x$ for all $x \in S$ (reflexivity)
2. $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity)
3. $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry)

Note: Some elements may be incomparable.

Example. Let S be the set of finite groups up to isomorphism. Define $H \leq G$ if H is a subgroup of G. Then S is a poset.

Example: $\mathbb{Z}_2 \leq \mathbb{Z}_4$, but S_3 and $\mathbb{Z}_6$ are not comparable.

Example. Let 2^S be the power set of a set S, ordered by inclusion:

$$X \leq Y \iff X \subseteq Y$$

This makes 2^S into a poset.

Definition (Totally Ordered Subset)**.** A subset $E \subseteq S$ of a poset $(S, \leq)$ is **totally ordered** if for all $x, y \in E$, either $x \leq y$ or $y \leq x$.

Example.

- $(\mathbb{R}, \leq)$ is totally ordered.
- 2^S and the group poset from earlier are not totally ordered.

Theorem (Zorn's Lemma)**.** Let S be a poset. If every totally ordered subset $E \subseteq S$ has an upper bound in S (i.e., there exists $x \in S$ such that $y \leq x$ for all $y \in E$), then S has a maximal element $z \in S$; that is,

$$z \leq y \Rightarrow z = y.$$

Example. Every vector space V over a field $\mathbb{F}$ has a basis.

Proof. Let $\mathcal{S}$ be the set of all linearly independent subsets of V, partially ordered by inclusion.

Step 1: Let $\mathcal{T} \subseteq \mathcal{S}$ be a totally ordered subset. Define

$$B := \bigcup_{T \in \mathcal{T}} T.$$

We claim that B is linearly independent.

Proof of claim: Suppose $v_1, \ldots, v_k \in B$. Then each $v_i \in T_i$ for some $T_i \in \mathcal{T}$. Since $\mathcal{T}$ is totally ordered, there exists a maximal element $T_j \in \mathcal{T}$ such that $T_i \subseteq T_j$ for all i.

Then $\{v_1, \ldots, v_k\} \subseteq T_j$, and since $T_j \in \mathcal{S}$, it is linearly independent. Thus, any linear combination

$$\sum_{i=1}^{k} a_i v_i = 0 \quad \Rightarrow \quad a_i = 0 \text{ for all } i.$$

So B is linearly independent. Hence, $B \in \mathcal{S}$, and B is an upper bound for $\mathcal{T}$.

Step 2: By Zorn's Lemma, $\mathcal{S}$ has a maximal element $Z \in \mathcal{S}$. That is, Z is linearly independent, and for all $Y \in \mathcal{S}$, if $Z \subseteq Y$, then $Z = Y$.

We claim Z is a basis for V.

Proof of claim: $Z \in \mathcal{S}$, so it is linearly independent. Suppose Z does not span V; then there exists $v \in V$ such that $v \notin \operatorname{span}(Z)$. Then $Z \cup \{v\}$ is linearly independent (since v is not in the span), contradicting the maximality of Z.

Therefore, Z spans V, and is a basis. $\square$

Tychonoff's Theorem

Theorem (Alexander Subbasis Theorem)**.** Let $(Y, \mathcal{T})$ be a topological space, and let $\mathcal{S} \subseteq \mathcal{T}$ be a subbasis for the topology $\mathcal{T}$.

Suppose that every open cover of Y by elements of $\mathcal{S}$ has a finite subcover. Then Y is compact.

In other words, if the finite subcover property holds for a subbasis of $\mathcal{T}$, then it holds for all open covers in $\mathcal{T}$.

Theorem (Tychonoff's Theorem)**.** Any product of compact topological spaces is compact in the product topology.

2 Algebraic Topology

Definition (Group)**.** A **group** G is a set together with two maps:

- a **multiplication** map $m : G \times G \to G$, denoted $(a, b) \mapsto ab$
- an **inverse** map $i : G \to G$, denoted $a \mapsto a^{-1}$

such that the following axioms hold:

1. **(Associativity)** For all $a, b, c \in G$, we have
$$(ab)c = a(bc)$$
2. **(Identity)** There exists an element $e \in G$ such that for all $a \in G$,
$$ae = a = ea$$
3. **(Inverse)** For all $a \in G$, there exists $a^{-1} \in G$ such that
$$aa^{-1} = e = a^{-1}a$$

Remark. It follows from the axioms that the identity element $e \in G$ is unique, and for each $a \in G$, the inverse a^{-1} is also unique.

Definition (Abelian Group)**.** A group G is called **abelian** if for all $a, b \in G$, we have
$$ab = ba$$

In this case, we often write the group operation using addition instead of multiplication.

Definition (Subgroup)**.** A subset $H \subseteq G$ of a group G is a **subgroup** if H is itself a group under the operation inherited from G. That is:

- $e_G \in H$
- $a, b \in H \Rightarrow ab \in H$
- $a \in H \Rightarrow a^{-1} \in H$

Example.
$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$
are subgroups under addition.

Definition (Group Homomorphism)**.** If G and H are groups, a function $f : G \to H$ is a **group homomorphism** if
$$f(ab) = f(a)f(b) \quad \text{for all } a, b \in G$$

Remark. A group homomorphism $f : G \to H$ satisfies:
$$f(e_G) = e_H, \quad f(a^{-1}) = f(a)^{-1}$$

Definition (Group Isomorphism)**.** A group homomorphism $f : G \to H$ is an **isomorphism** if f is both one-to-one and onto.

Homotopies of Paths

Definition (Path)**.** Let X be a topological space. A **path** γ in X from a to b is a continuous map

$$\gamma : [0,1] \to X \quad \text{such that } \gamma(0) = a \text{ and } \gamma(1) = b$$

Remark. There is a distinction between the map γ and its image $\mathrm{Im}(\gamma)$; γ might traverse the image at different speeds or directions.

Definition (Homotopic Paths)**.** Let $\gamma_0, \gamma_1 : [0,1] \to X$ be two paths from a to b. We say that γ_0 and γ_1 are **homotopic relative to endpoints** if there exists a continuous map

$$F : [0,1] \times [0,1] \to X$$

such that:

1. $\gamma_0(s) = F(s,0) \quad \forall s \in [0,1]$
2. $\gamma_1(s) = F(s,1) \quad \forall s \in [0,1]$
3. $F(0,t) = a, \quad F(1,t) = b \quad \forall t \in [0,1]$

In this case, we say F is a **homotopy from** γ_0 **to** γ_1.

Proposition. Homotopy relative to endpoints is an equivalence relation on paths.

$$\gamma_0 \simeq \gamma_1 \quad \text{is an equivalence relation.}$$

Proof. We check the three properties:

1. **Reflexivity:** Any path γ is homotopic to itself via
$$F(s,t) = \gamma(s)$$
which is clearly continuous.
2. **Symmetry:** If $\gamma_0 \simeq \gamma_1$ via $F(s,t)$, then define
$$G(s,t) = F(s, 1-t)$$
to obtain a homotopy $\gamma_1 \simeq \gamma_0$.
3. **Transitivity:** If $\gamma_0 \simeq \gamma_1$ via F, and $\gamma_1 \simeq \gamma_2$ via G, then define
$$H(s,t) = \begin{cases} F(s,2t), & 0 \leq t \leq \frac{1}{2} \\ G(s,2t-1), & \frac{1}{2} < t \leq 1 \end{cases}$$
This gives a continuous homotopy $\gamma_0 \simeq \gamma_2$.

∎

Remark. If $X_1, X_2 \subseteq X$ are closed, $X_1 \cup X_2 = X$, and $H|_{X_1}$, $H|_{X_2}$ are continuous, then H is continuous on all of X.

Definition (Path Concatenation)**.** Let α be a path from a to b, and β a path from b to c. The **concatenation** $\alpha \cdot \beta : [0,1] \to X$ is defined by

$$(\alpha \cdot \beta)(s) := \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

Lemma. If $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$, then

$$\alpha_0 \cdot \beta_0 \simeq \alpha_1 \cdot \beta_1$$

Proof. Let $F : [0,1] \times [0,1] \to X$ be a homotopy from α_0 to α_1, and let $G : [0,1] \times [0,1] \to X$ be a homotopy from β_0 to β_1.

Define a new homotopy $H : [0,1] \times [0,1] \to X$ by

$$H(s,t) := \begin{cases} F(2s,t), & 0 \le s \le \frac{1}{2} \\ G(2s-1,t), & \frac{1}{2} \le s \le 1 \end{cases}$$

Then H is a homotopy from $\alpha_0 \cdot \beta_0$ to $\alpha_1 \cdot \beta_1$, as required. □

Lemma (Reparameterization Trick)**.** Let $\varphi : [0,1] \to [0,1]$ be any continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then the path φ is homotopic (rel endpoints) to the identity path $s \mapsto s$.

Proof. Define the homotopy $H : [0,1] \times [0,1] \to [0,1]$ by

$$H(s,t) = \varphi(s) + t(s - \varphi(s))$$

This is a straight-line homotopy between φ and the identity map.

We verify:

$$\begin{aligned} H(s,0) &= \varphi(s) \\ H(s,1) &= s \\ H(0,t) &= \varphi(0) + t(0 - \varphi(0)) = 0 \\ H(1,t) &= \varphi(1) + t(1 - \varphi(1)) = 1 \end{aligned}$$

and H is continuous, as it is built from continuous functions.

Thus, $\varphi \simeq \mathrm{id}_{[0,1]}$ rel endpoints. □

Lemma. Let α be a path from a to b. Then

$$e_a \cdot \alpha \simeq \alpha \simeq \alpha \cdot e_b$$

where e_a and e_b are the constant paths at a and b, respectively.

Proof. We show $e_a \cdot \alpha \simeq \alpha$. The other case is similar.

Define $\varphi : [0,1] \to [0,1]$ by

$$\varphi(s) = \begin{cases} 0, & 0 \le s \le \frac{1}{2} \\ 2s-1, & \frac{1}{2} \le s \le 1 \end{cases}$$

Then the concatenated path $e_a \cdot \alpha$ is just $\alpha \circ \varphi$, and by the reparameterization lemma, this is homotopic to α.

Hence $e_a \cdot \alpha \simeq \alpha$. □

Lemma. Let α be a path from a to b. Then

$$\alpha \cdot \alpha^{-1} \simeq e_a \quad \text{and} \quad \alpha^{-1} \cdot \alpha \simeq e_b$$

where $\alpha^{-1}(s) := \alpha(1-s)$ is the reverse path.

Proof. We construct a homotopy $H : [0,1] \times [0,1] \to X$ from $\alpha \cdot \alpha^{-1}$ to the constant path e_a, defined by:

$$H(s,t) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{t}{2} \\ \alpha(t), & \frac{t}{2} \leq s \leq 1 - \frac{t}{2} \\ \alpha(1-2s), & 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$

At $t = 0$, this is the path $\alpha \cdot \alpha^{-1}$, and at $t = 1$, we get the constant path $\alpha(1-s)$ concatenated with $\alpha(s)$, which cancels to the point a.

Hence, $\alpha \cdot \alpha^{-1} \simeq e_a$. A symmetric argument shows $\alpha^{-1} \cdot \alpha \simeq e_b$. □

2.1 Fundamental Group

Let X be a topological space, and let $x_0 \in X$ be a chosen **base point**. The pair (X, x_0) is called a **pointed topological space**.

Definition (Loop)**.** A **loop based at** x_0 is a continuous map

$$\gamma : [0,1] \to X \quad \text{such that} \quad \gamma(0) = x_0 = \gamma(1)$$

Definition (Fundamental Group)**.** The **fundamental group** $\pi_1(X, x_0)$ is the set of homotopy classes of loops based at x_0, where two loops are considered equivalent if they are homotopic relative to endpoints.

Theorem. The fundamental group $\pi_1(X, x_0)$ is a group, with multiplication given by composition (concatenation) of loops.

Proof. Let α and β be loops based at x_0. Then $\alpha \cdot \beta$ is again a loop based at x_0.

We previously showed that the operation

$$[\alpha][\beta] := [\alpha \cdot \beta]$$

is well-defined on homotopy classes.

We now verify the group axioms:

1. **Associativity:** For any loops α, β, γ,

   $$([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma])$$

   because path concatenation is associative up to homotopy.

2. **Identity:** The constant loop e_{x_0} satisfies

   $$[\alpha][e_{x_0}] = [\alpha] = [e_{x_0}][\alpha]$$

3. **Inverses:** The inverse loop $\alpha^{-1}(s) = \alpha(1-s)$ satisfies

   $$[\alpha][\alpha^{-1}] = [e_{x_0}] = [\alpha^{-1}][\alpha]$$

Thus, $\pi_1(X, x_0)$ is a group. $\square$

Example. We have

$$\pi_1(\mathbb{R}^n, \vec{0}) = \{[\vec{0}]\}$$

That is, the fundamental group of $\mathbb{R}^n$ at the origin is the trivial group.

Proof. Let $[\gamma] \in \pi_1(\mathbb{R}^n, \vec{0})$, so γ is a loop based at $\vec{0}$.
Define the homotopy $F : [0, 1] \times [0, 1] \to \mathbb{R}^n$ by

$$F(s, t) = t \cdot \gamma(s)$$

Then F is a homotopy from the constant loop $\vec{0}$ to γ, so $[\gamma] = [\vec{0}]$. $\square$

Definition (Convex Set)**.** A set $X \subset \mathbb{R}^n$ is **convex** if for any $\vec{x}, \vec{y} \in X$, the line segment

$$\{t\vec{x} + (1 - t)\vec{y} : t \in [0, 1]\} \subset X$$

That is, X contains the entire line segment between any two of its points.

Example. If $X \subset \mathbb{R}^n$ is convex, then

$$\pi_1(X, x_0) = \{[e_{x_0}]\}$$

Proof. Let γ be any loop based at x_0. Define

$$F(s, t) = (1 - t)x_0 + t \cdot \gamma(s)$$

This defines a homotopy from the constant loop e_{x_0} to γ, so

$$[\gamma] = [e_{x_0}]$$

and the fundamental group is trivial. $\square$

Theorem (Change of Basepoint)**.** Let α be a path from x_1 to x_0. Then there is an isomorphism

$$\pi_1(X, x_1) \cong \pi_1(X, x_0)$$

given by

$$f([\gamma]) = [\alpha \cdot \gamma \cdot \alpha^{-1}]$$

Proof. **(1) Well-definedness:** Suppose $\gamma \simeq \gamma'$. Then

$$\alpha \cdot \gamma \cdot \alpha^{-1} \simeq \alpha \cdot \gamma' \cdot \alpha^{-1}$$

by concatenation and homotopy compatibility. So $f([\gamma]) = f([\gamma'])$.
(2) Homomorphism: Let γ_1, γ_2 be loops based at x_1. Then:

$$\begin{aligned} f([\gamma_1][\gamma_2]) &= f([\gamma_1 \cdot \gamma_2]) \\ &= [\alpha \cdot (\gamma_1 \cdot \gamma_2) \cdot \alpha^{-1}] \\ &= [\alpha \cdot \gamma_1 \cdot \alpha^{-1}] \cdot [\alpha \cdot \gamma_2 \cdot \alpha^{-1}] \\ &= f([\gamma_1])f([\gamma_2]) \end{aligned}$$

(3) Isomorphism: Define the inverse

$$f^{-1} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{by} \quad [\delta] \mapsto [\alpha^{-1} \cdot \delta \cdot \alpha]$$

Then:

$$f(f^{-1}([\delta])) = f([\alpha^{-1} \cdot \delta \cdot \alpha]) = [\alpha \cdot \alpha^{-1} \cdot \delta \cdot \alpha \cdot \alpha^{-1}] = [\delta]$$
$$f^{-1}(f([\gamma])) = f^{-1}([\alpha \cdot \gamma \cdot \alpha^{-1}]) = [\alpha^{-1} \cdot \alpha \cdot \gamma \cdot \alpha^{-1} \cdot \alpha] = [\gamma]$$

Thus, f is an isomorphism. □

Definition. If $f : X \to Y$ is a continuous map between topological spaces, and $y_0 = f(x_0)$, we write

$$f : (X, x_0) \to (Y, y_0)$$

Theorem. Given a continuous map $f : (X, x_0) \to (Y, y_0)$, there is an **induced homomorphism**

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

defined by

$$f_*([\alpha]) := [f \circ \alpha]$$

This map sends the homotopy class of a loop α in X based at x_0 to the homotopy class of the loop $f \circ \alpha$ in Y based at y_0.

Theorem. Let $f : X \to Y$ be a continuous map between topological spaces, and suppose $x_0 \in X$ is a base point with $y_0 = f(x_0)$. Then f induces a homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

defined by

$$[\alpha] \mapsto [f \circ \alpha],$$

where $(f \circ \alpha)(s) = f(\alpha(s))$.

Proposition (Functorality - Identity)**.** Given the identity map $\mathrm{id}_{(X,x_0)} : (X, x_0) \to (X, x_0)$, the induced homomorphism

$$(\mathrm{id})_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is the identity map:

$$[\alpha] \mapsto [\alpha].$$

Proposition (Functorality - Composition)**.** Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous basepoint-preserving maps. Then the composition

$$(g \circ f) : (X, x_0) \to (Z, z_0)$$

induces a homomorphism

$$(g \circ f)_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$$

given by

$$(g \circ f)_* = g_* \circ f_*.$$

Corollary. If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then the induced map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism of groups.

Proof. Since f is a homeomorphism, it has an inverse f^{-1}. The composition laws for induced homomorphisms give:

$$(f^{-1} \circ f)_* = (f^{-1})_* \circ f_* = \mathrm{id}_{\pi_1(X,x_0)},$$
$$(f \circ f^{-1})_* = f_* \circ (f^{-1})_* = \mathrm{id}_{\pi_1(Y,y_0)}.$$

Thus f_* is an isomorphism of groups with inverse $(f^{-1})_*$. □

Definition (Simply Connected)**.** If X is path-connected and $\pi_1(X, x_0)$ is trivial, then X is **simply connected**.

Homotopy of Maps

Definition. Let $f_0, f_1 : X \to Y$ be continuous maps. We say that f_0 is **homotopic** to f_1, written $f_0 \simeq f_1$, if there exists a homotopy:

$$F : X \times [0,1] \to Y \quad \text{(continuous)}$$

such that

$$F(x, 0) = f_0(x), \quad F(x, 1) = f_1(x).$$

We often write $F(x, t) = f_t(x)$.

Definition. If, in addition, there exists a subset $A \subset X$ such that

$$f_0|_A = f_1|_A,$$

we say $f_0 \simeq f_1$ **rel** A. That is, there exists a homotopy $F : X \times [0,1] \to Y$ such that

$$F(a, t) = f_0(a) = f_1(a) \quad \text{for all } a \in A,\ t \in [0,1].$$

Example. If $A = \{0, 1\} \subset [0, 1]$, this describes homotopies *rel endpoints* for paths.

Theorem. If $f_t : X \to Y$ is a homotopy from f_0 to f_1, then

$$(f_0)_* = \varphi_\alpha \circ (f_1)_*,$$

where $\alpha(t) = f_t(x_0)$ is the *track* of the basepoint, and

$$\varphi_\alpha : \pi_1(Y, f_1(x_0)) \to \pi_1(Y, f_0(x_0))$$

is defined by

$$[\gamma] \mapsto [\alpha \cdot \gamma \cdot \alpha^{-1}].$$

Proof. Let $\alpha(t) = f_t(x_0)$, so α is a path in Y from $f_0(x_0)$ to $f_1(x_0)$. Let $\alpha^t(s) = \alpha(ts)$, a reparameterization.

We show that $\alpha^t \cdot (f_t \circ \gamma) \cdot (\alpha^t)^{-1}$ is a homotopy from $f_0 \circ \gamma$ to $f_1 \circ \gamma$, relative to endpoints (assuming $\gamma(0) = \gamma(1) = x_0$).

Hence, the homotopy classes of $f_0 \circ \gamma$ and $\alpha \cdot (f_1 \circ \gamma) \cdot \alpha^{-1}$ are equal in $\pi_1(Y, f_0(x_0))$, i.e.,

$$(f_0)_*([\gamma]) = \varphi_\alpha((f_1)_*([\gamma])).$$

□

Definition. We say a continuous map $g : Y \to X$ is a **homotopy inverse** of a continuous function $f : X \to Y$ if

$$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y.$$

In this situation, we say f is a **homotopy equivalence**, and that X and Y are **homotopy equivalent spaces**.

Example. $\mathbb{R}^n$ is homotopy equivalent to $*$ (a one-point space).

$$\begin{aligned} f &: * \to \mathbb{R}^n, & * &\mapsto \vec{0}, \\ g &: \mathbb{R}^n \to *, & x &\mapsto *. \end{aligned}$$

1. $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n}$
2. $g \circ f = \mathrm{id}_*$

Proof. For (2), we have $g \circ f = \mathrm{id}_*$, so the identity holds strictly.

For (1), note that

$$f \circ g(x) = \vec{0}, \quad \text{for all } x \in \mathbb{R}^n.$$

Define a homotopy $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by

$$F(x,t) = x \cdot t.$$

It is clear that

$$F(x,0) = f \circ g(x), \qquad F(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x),$$

so $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n}$. □

Definition. We say a topological space X is **contractible** if it is homotopy equivalent to a point.

Example.

- $\mathbb{R}^n$ is contractible.
- The closed unit ball in $\mathbb{R}^n$, denoted

  $$\overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\},$$

  is also contractible, as is its interior.

Example. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then S^1 is **not contractible**, because it is homotopy equivalent to the punctured complex plane $\mathbb{C} \setminus \{0\}$, which is not simply connected.

2.2 Covering Spaces

Definition (Covering Space)**.** Let X be a topological space. A *covering space* for X is a pair (E, π), where E is a topological space and $\pi : E \to X$ is a continuous surjective map such that for every $x \in X$, there exists an open neighborhood U_x of x such that

$$\pi^{-1}(U_x) = \bigsqcup_{\alpha} V_\alpha$$

is a disjoint union of open sets $V_\alpha \subset E$, each of which is mapped homeomorphically onto U_x by π, i.e.,

$$\pi|_{V_\alpha} : V_\alpha \to U_x$$

is a homeomorphism for each α.

Definition (Evenly Covered Neighborhood)**.** Let $p : E \to X$ be a continuous map between topological spaces. An open subset $U \subseteq X$ is said to be *evenly covered* by p if

$$p^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha$$

is a disjoint union of open subsets $V_\alpha \subseteq E$, each of which is mapped homeomorphically onto U by p, i.e., the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for each $\alpha \in A$.

Definition (Covering Map)**.** A continuous surjection $p : E \to X$ is called a *covering map* if every point $x \in X$ has an open neighborhood U such that U is evenly covered by p. In this case, E is called a *covering space* over X.

Definition (Fiber)**.** Let $p : E \to X$ be a map. For $x \in X$, the set

$$p^{-1}(x)$$

is called the *fiber* over x.

Definition (Sheets of an Evenly Covered Neighborhood)**.** Let $p : E \to X$ be a covering map, and let $U \subseteq X$ be an open set that is evenly covered by p. The open subsets $V_\alpha \subseteq p^{-1}(U)$ such that each restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism are called the *sheets* of $p^{-1}(U)$.

Proposition. Let $p : E \to X$ be a covering map, and suppose $U \subseteq X$ is an open, connected, and evenly covered neighborhood. Then the sheets of $p^{-1}(U)$ are precisely the connected components of $p^{-1}(U)$.

Definition (Lift)**.** Let $\pi : E \to X$ be a covering space and let $f : Y \to X$ be a continuous map. A continuous map $\tilde{f} : Y \to E$ is called a *lift* of f if $\pi \circ \tilde{f} = f$.

Theorem (Homotopy Lifting)**.** Suppose $\pi : E \to X$ is a covering space and $f_t : Y \to X$ is a homotopy for $t \in [0, 1]$. If $\tilde{f}_0 : Y \to E$ is a lift of f_0, then there exists a unique lift $\tilde{f}_t : Y \to E$ of the entire homotopy f_t.

Theorem (Path Lifting)**.** Let $\pi : E \to X$ be a covering space. Let $\gamma : [0, 1] \to X$ be a path starting at $x_0 \in X$. Then, given a point $\tilde{x}_0 \in \pi^{-1}(\{x_0\})$, there exists a unique lift $\tilde{\gamma} : [0, 1] \to E$ of γ such that $\tilde{\gamma}(0) = \tilde{x}_0$.

Definition. Let $\pi : E \to X$ be a covering space, and let $x_0 \in X$ and $\tilde{x}_0 \in \pi^{-1}(x_0)$. We define a function

$$\Phi : \pi_1(X, x_0) \to \pi^{-1}(x_0)$$

by sending the homotopy class $[\gamma]$ to the endpoint $\tilde{\gamma}(1)$ of the unique lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$.

Lemma. The function Φ is well-defined.

Proof. Given a path γ, the path lifting theorem guarantees the existence and uniqueness of a lift $\tilde{\gamma}$ starting at $\tilde{x}_0$, so the value $\tilde{\gamma}(1)$ is well-defined.

To show independence of representative: suppose $\gamma_0 \simeq \gamma_1$ rel endpoints. Then there exists a homotopy $F : [0,1] \times [0,1] \to X$ from γ_0 to γ_1 such that

$$F(s,t) = \gamma_t(s), \quad \text{with } F(0,t) = F(1,t) = x_0.$$

By the homotopy lifting property, this homotopy lifts to $\tilde{F}$ starting at $\tilde{x}_0$, which implies that

$$\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1).$$

Therefore, $\Phi([\gamma])$ is independent of the representative path, and the function is well-defined. $\square$

Fact (Facts about Covering Spaces)**.**

- Any sufficiently nice topological space X has a simply connected covering space

  $$\widetilde{X} \xrightarrow{\pi} X,$$

  which is unique up to isomorphism. This space $\widetilde{X}$ is called the *universal cover* of X.

- There is a correspondence between connected covering spaces of X (up to isomorphism of covering maps) and subgroups of the fundamental group $\pi_1(X)$, given by:

  $$\left\{ \begin{array}{c} \text{Connected covering spaces} \\ \text{of } X \text{ (up to equivalence)} \end{array} \right\} \longleftrightarrow \left\{ \text{ Subgroups } H \subseteq \pi_1(X) \right\}.$$

  The universal cover corresponds to the trivial subgroup $\{e\}$, and the identity covering $X \to X$ corresponds to $\pi_1(X)$ itself. Intermediate covers correspond to intermediate subgroups $H \subseteq \pi_1(X)$.

Theorem (Brouwer Fixed Point Theorem)**.** Let $D^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ denote the closed n-dimensional disk, with boundary $\partial D^n = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\} = S^{n-1}$.

Then any continuous map $f : D^n \to D^n$ has a fixed point. That is, there exists $z \in D^n$ such that

$$f(z) = z.$$

Theorem (Fundamental Theorem of Algebra)**.** Let $p : \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial, i.e.,

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

Then $p(z)$ has at least one zero in $\mathbb{C}$.