Math 115AH Notes

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Key Definitions/Axioms

Definition (Field Axioms)**.** A field F is a set with two (binary–meaning two inputs of same type) operations $+$ and $\cdot$ (called addition and multiplication, respectively) such that for each pair of elements $x, y \in F$, there are unique elements in F, denoted $x+y$ and $x \cdot y$, satisfying the following conditions for all elements $a, b, c \in F$:

(i) $a + b = b + a$ and $a \cdot b = b \cdot a$
(Commutativity of addition and multiplication)

(ii) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(Associativity of addition and multiplication)

(iii) There exist distinct elements 0 and 1 in F such that
$0 + a = a$ and $1 \cdot a = a$
(Existence of identity elements for addition and multiplication)

(iv) For each element $a \in F$ and each nonzero element $b \in F$, there exist elements c and $d \in F$ such that
$a + c = 0$ and $b \cdot d = 1$
(Existence of inverses for addition and multiplication)

(v) $a \cdot (b + c) = a \cdot b + a \cdot c$
(Distributivity of multiplication over addition)

The elements $x + y$ and $x \cdot y$ are called the sum and product, respectively, of x and y. The elements 0 (read "zero") and 1 (read "one") are the additive and multiplicative identity elements, respectively.

Definition (Relation)**.** A *relation* on a set A is a subset of the Cartesian product $A \times A$. For elements $a, b \in A$, if the pair (a, b) is in this subset, we write $a \sim b$ and say that a is related to b.

Definition (Equivalence Relation)**.** An *equivalence relation* on a set A is a relation denoted by $\sim$ that satisfies the following three properties for all $a, b, c \in A$:

1. *Reflexivity*: $a \sim a$ for all $a \in A$.
2. *Symmetry*: If $a \sim b$, then $b \sim a$.
3. *Transitivity*: If $a \sim b$ and $b \sim c$, then $a \sim c$.

If $\sim$ is an equivalence relation on A, and $a \sim b$, we say that a is equivalent to b.

Definition (Equivalence Class)**.** Given an equivalence relation R on a set S and an element $a \in S$, let $[a] = \{b \in S \mid a \sim b\}$. The set $[a]$ is called the *equivalence class* of the element a. Note that $[a]$ is a subset of S, consisting of all elements of S which are related to a under the equivalence relation R.

Definition (Quotient of a Relation)**.** Given a set S with an equivalence relation R, define the quotient set S/R as

$$S/R = \{[a] \mid a \in S\}.$$

We refer to S/R as the quotient of S by the relation R.

Definition (Integers Modulo n)**.** Recall that $\mathbb{Z}/n\mathbb{Z}$ is the set $\{[0], [1], [2], \ldots, [n-1]\}$. We can define two operations on this set, as follows:

Addition: An operation $+_n : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, given on $a, b \in \mathbb{Z}/n\mathbb{Z}$ by

$$[a] + [b] = [a + b],$$

where the $+$ on the right-hand side is the usual addition in $\mathbb{Z}$.

Multiplication: An operation $\cdot_n : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, given on $a, b \in \mathbb{Z}/n\mathbb{Z}$ by

$$[a] \cdot [b] = [a \cdot b],$$

where the $\cdot$ on the right-hand side is the usual multiplication in $\mathbb{Z}$.

Definition (Vector Space)**.** A vector space V over a field $\mathbb{F}$ is a collection of objects called vectors, along with operations of vector addition and scalar multiplication that satisfy the following eight axioms.

(VS 1) For all $\vec{x}, \vec{y}$ in V, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity of addition).

(VS 2) For all $\vec{x}, \vec{y}, \vec{z}$ in V, $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ (associativity of addition).

(VS 3) There exists an element in V denoted by 0 such that $\vec{x} + 0 = \vec{x}$ for each $\vec{x}$ in V.

(VS 4) For each element $\vec{x}$ in V there exists an element $\vec{y}$ in V such that $\vec{x} + \vec{y} = 0$.

(VS 5) For each element $\vec{x}$ in V, $1\vec{x} = \vec{x}$.

(VS 6) For each pair of elements a, b in F and each element $\vec{x}$ in V, $(ab)\vec{x} = a(b\vec{x})$.

(VS 7) For each element a in F and each pair of elements $\vec{x}, \vec{y}$ in V, $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$.

(VS 8) For each pair of elements a, b in F and each element $\vec{x}$ in V, $(a + b)\vec{x} = a\vec{x} + b\vec{x}$.

Definition (Linearly Dependent)**.** A subset S of a vector space V is called *linearly dependent* if there exist a finite number of distinct vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = \vec{0}.$$

In this case, we also say that the vectors of S are linearly dependent.

Definition (Linearly Independent)**.** A subset S of a vector space V is called *linearly independent* if the only scalars $a_1, a_2, \ldots, a_n$ that satisfy

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = \vec{0}$$

are $a_1 = a_2 = \cdots = a_n = 0$. In other words, the only representation of the zero vector as a linear combination of vectors in S is the trivial representation.

Definition (Linear Transformation)**.** A function $T : V \to W$ is a **linear transformation** from V to W if:

1. For all $x, y \in V$, $T(x + y) = T(x) + T(y)$.

2. For all $x \in V$ and $\lambda \in \mathbb{F}$, $T(\lambda x) = \lambda T(x)$.

Definition (Function)**.** A **function** f is the data of:

1. a set A called the **domain**,
2. a set B called the **codomain**,
3. a rule or formula that associates to each element in the domain an element in the codomain.

Definition (Kernel and Image)**.** Let $T : V \to W$ be linear.

- The kernel, or null space, of T is

$$\ker(T) := \{v \in V \mid T(v) = \vec{0}_W\}.$$

- The image, or range, of T is

$$\operatorname{Im}(T) := \{w \in W \mid \exists v \in V : T(v) = w\}.$$

Definition (Linear Combination)**.** Let S be a subset of a vector space V over a field $\mathbb{F}$.

- A **linear combination** of vectors in S is any finite sum $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \sum_{i=1}^n a_i\vec{v}_i$, where $a_i \in \mathbb{F}$ and $\vec{v}_i \in S$.
- The set of all linear combinations of vectors in S is called the **span** of S, written as $\operatorname{span}(S)$.

Definition (Basis)**.** A **basis** for a vector space V over $\mathbb{F}$ is a set $\mathcal{B} \subseteq V$ such that:

1. $\mathcal{B}$ is linearly independent.
2. $\operatorname{span}(\mathcal{B}) = V$.

We say $\mathcal{B}$ **spans** or **generates** V.

Definition (Left Multiplication Transformation)**.** Given a matrix $A \in \operatorname{Mat}_{n\times m}(\mathbb{F})$, we let $L_A : \mathbb{F}^m \to \mathbb{F}^n$ be the linear transformation defined by

$$L_A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.$$

Definition (Invertibility)**.** Let X and Y be sets. A function $f : X \to Y$ is invertible if there is a function $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y.$$

Terminology: g is an *inverse* for f. We write $g = f^{-1}$.
Note: A function $f : X \to Y$ is invertible if and only if f is one-to-one and onto.

Definition (Matrix Invertibility)**.** A matrix $A \in \text{Mat}_{n\times n}(\mathbb{F})$ is invertible if there exists a matrix $B \in \text{Mat}_{n\times n}(\mathbb{F})$ such that

$$A \cdot B = B \cdot A = I_n.$$

Definition (Linear Isomorphism)**.** A linear transformation $T : V \to W$ is called an *isomorphism* if T is invertible.

If there exists an isomorphism $T : V \to W$, we say that V is isomorphic to W.

Example: $T : P_1(\mathbb{R}) \to \mathbb{R}^2$

$$T(a + bx) = (a, b)$$

Definition (Determinant)**.** Let $A \in \text{Mat}_{n\times n}(\mathbb{F})$. We recursively define:

- For $n = 1$: $\det(a) = a$.
- For $n = 2$:
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$
- For $n \geq 3$: For an $n \times n$ matrix A, fix $j \in \{1, \dots, n\}$. Then,
$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij}) a_{ij},$$
where A_{ij} is the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i-th row and j-th column.

Smaller Key Results

Definition (Span and L.D.)**.** Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Definition (Addition of Subspaces)**.** Let S, T be subsets of V. Let

$$S + T := \{v + v' \mid v \in S, v' \in T\}.$$

If $S = U_1$ and $T = U_2$ are subspaces, then so is $U_1 + U_2$.

Theorem (1.8 in Book)**.** Given a collection $\mathcal{B}$ of distinct vectors in V, $\mathcal{B}$ is a basis $\iff$ every vector in V can be written uniquely as a linear combination of vectors in $\mathcal{B}$.

Proposition (Determinant Properties)**.** We have the following properties of the determinant:

1. Let A be an $n \times n$ matrix with entries in $\mathbb{F}$. Then $\det(A) \neq 0$ if and only if A is invertible.

2. The definition of the determinant doesn't depend on the choice of $i \in \{1, \ldots, n\}$.

3. $\det(A) = \det(A^t)$.

4. $\det(AB) = \det(A) \cdot \det(B)$.

5. The function $\det : (\mathbb{F}^n)^{\times n} \to \mathbb{F}$ is linear in each argument.

6. Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$,
$$A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.$$
Let $A(i,j)$ denote the matrix obtained by swapping the i-th and j-th columns. Let $A^*(i,j,\lambda)$ denote the matrix obtained by adding $\lambda \vec{v}_j$ to $\vec{v}_i$. Then:
   - $\det(A(i,j)) = -\det(A)$
   - $\det(A^*(i,j,\lambda)) = \det(A)$

7. If A has a row or column that is all zeros, then $\det(A) = 0$.

Major Theorems

Theorem (The Replacement Theorem)**.** Let V be a vector space over a field $\mathbb{F}$.
If we are given sets $G, L \subseteq V$ such that:

- G has n elements and $\mathrm{span}(G) = V$,
- L has m elements and is linearly independent,

then:

- $m \leq n$,
- There exists $H \subseteq V$ with $n - m$ elements such that $L \cup H$ generates V.

Theorem (Corollaries to the Replacement Theorem)**.** Assume the sets are finite

1. Any two bases have the same number of elements.
2. We call the number of elements in a basis the *dimension* of V.
3. If $L \subseteq V$ is linearly independent, then $\#L \leq \dim(V)$. If $G \subseteq V$ spans V, then $\#G \geq \dim(V)$.
4. If $\mathrm{span}(S) = V$ and $\#S = \dim(V)$, then S is a basis for V. If L is linearly independent and $\#L = \dim(V)$, then L is a basis for V.
5. Every linearly independent set in V is contained in a basis. Every spanning set in V contains a basis.

Theorem (The Dimension Theorem)**.** For V, W vector spaces over $\mathbb{F}$, let $T : V \to W$ be a linear transformation.

$$\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(V)$$

where $\dim(\ker(T))$ is the nullity $n(T)$ and $\dim(\mathrm{Im}(T))$ is the rank $r(T)$.

Theorem (Linear Transformation Defined on Basis Vectors)**.** If $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is a basis for V and $\{\vec{w}_1, \ldots, \vec{w}_n\} \subset W$, then there is a unique linear transformation $T : V \to W$ such that $T(\vec{u}_i) = \vec{w}_i$ for all i.

Coordinate Representation and Change of Bases

Definition (Coordinate Representation of Vectors)**.** Given a basis $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ for V and $\vec{v} \in V$, the $\mathcal{B}$-coordinate representation, or $\mathcal{B}$-coordinate vector, for $\vec{v}$ is the column vector

$$[\vec{v}]_{\mathcal{B}} := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n,$$

where $a_i \in \mathbb{F}$ are the unique scalars such that

$$\vec{v} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n.$$

Theorem (Coordinate Transformation)**.** Let $T : V \to W$ be linear, with $\dim(V)$ and $\dim(W)$ finite. Let β be a basis for V and γ a basis for W. Then

$$[T]_\beta^\gamma [\vec{v}]_\beta = [T(\vec{v})]_\gamma.$$

Theorem (Property of Coordinate Matrices)**.** Let V, W, Z be vector spaces over $\mathbb{F}$, with bases β, γ, δ. Let $T : V \to W$ and $H : W \to Z$ be linear. Then

$$[H \circ T]_\beta^\delta = [H]_\gamma^\delta [T]_\beta^\gamma.$$

Theorem (Properties of Coordinate Matrices (2.8))**.** Suppose $T_1, T_2 : V \to W$ are linear, with bases β for V and γ for W. Then:

1. $[T_1 + T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$
2. For all $\lambda \in \mathbb{F}$, $[\lambda T_1]_\beta^\gamma = \lambda [T_1]_\beta^\gamma$
3. $[\mathrm{id}_W]_\beta^\beta = I_n$, where $n = \dim(V)$

Theorem (Equivalent Statement to Transformation Equality)**.** Let $T : V \to W$ and $S : V \to W$ be linear. Let β be a finite basis for V and γ a finite basis for W. Then $T = S$ if and only if

$$[T]_\beta^\gamma = [S]_\beta^\gamma.$$

Theorem (Matrices and Linear Transformations)**.** Let $A \in \text{Mat}_{m\times n}(\mathbb{F})$. Let β_1, β_2 be the standard bases for $\mathbb{F}^n$ and $\mathbb{F}^m$ (respectively). Then:

1. $[L_A]_{\beta_1}^{\beta_2} = A$
2. For all $B \in \text{Mat}_{l\times m}(\mathbb{F})$,
$$L_B \circ L_A = L_{B\cdot A}$$

Theorem (Theorems on Invertibility)**.** We have the following theorems

a A matrix $A \in \text{Mat}_{n\times n}(\mathbb{F})$ is invertible if and only if $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is invertible and $(L_A)^{-1} = L_{A^{-1}}$.

b Suppose V, W are finite-dimensional, β is a basis for V, and γ is a basis for W. Then $T : V \to W$ is invertible if and only if $[T]_\beta^\gamma$ is invertible. If T is invertible, then
$$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}.$$

Lemma (On Coordinate Matrix Properties)**.** Let $A \in \text{Mat}_{m\times n}(\mathbb{F})$, $B \in \text{Mat}_{l\times m}(\mathbb{F})$. Let $\beta_1, \beta_2, \beta_3$ be the standard bases for $\mathbb{F}^n$, $\mathbb{F}^m$, $\mathbb{F}^l$ (respectively). Then:

1. $[L_A]_{\beta_1}^{\beta_2} = A$
2. $L_{B\cdot A} = L_B \circ L_A$

Definition (Change of Coordinate Matrix)**.** Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let β, β' be two bases for V. The matrix
$$Q = [\text{Id}_V]_{\beta'}^\beta$$
is a change of coordinates matrix that changes β'-coordinates into β-coordinates.

Definition (Notational Note)**.** Let $T : V \to V$. If β, β' are both finite bases for V, then:

$$[T]_\beta = [T]_\beta^\beta,$$
$$[T]_{\beta'} = [T]_{\beta'}^{\beta'}.$$

Theorem (Theorems on Change of Coordinate Matrices)**.** We have the following:

1. Q is invertible and $Q^{-1} = [\text{id}_V]_\beta^{\beta'}$.
2. For every $\vec{v} \in V$,
$$[\vec{v}]_\beta = Q[\vec{v}]_{\beta'}.$$

3. $Q^{-1}[T]_\beta Q = [T]_{\beta'}$.

Definition (Similar Matrices)**.** Matrices $A, B \in \text{Mat}_{n\times n}(\mathbb{F})$ are *similar* if there exists an invertible matrix $Q \in \text{Mat}_{n\times n}(\mathbb{F})$ such that

$$A = Q^{-1}BQ.$$

We also have the following facts:

- If $A = [T]_\beta$ and $B = [T]_\gamma$ for some $T : V \to V$, $\dim V = n$, then A is similar to B.
- If A is similar to B, then $\det(A) = \det(B)$.
- If A is similar to B, then B is similar to A.

Eigenvalues, Eigenvectors, and Diagonalizability

Definition (Eigenvalue/Eigenvector)**.** Given $\vec{v} \in V$ nonzero and $T : V \to V$, $\vec{v}$ is an *eigenvector* of T if there exists $\lambda \in \mathbb{F}$ such that

$$T(\vec{v}) = \lambda\vec{v}.$$

λ is called an *eigenvalue* for T.

Definition (Eigenspace)**.** Let $\lambda \in \mathbb{F}$ be an eigenvalue of $T : V \to V$. Then the λ-eigenspace of T is the subset

$$E_\lambda := \{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\} \subseteq V.$$

In other words, it is the set of all eigenvectors with eigenvalue λ union $\{\vec{0}\}$.

If λ is an eigenvalue of $A \in \text{Mat}_{n\times n}(\mathbb{F})$, then the λ-eigenspace of A is the λ-eigenspace of $L_A : \mathbb{F}^n \to \mathbb{F}^n$. In other words, the λ-eigenspace of A is

$$\{\vec{v} \in \mathbb{F}^n \mid L_A(\vec{v}) = \lambda\vec{v}\}.$$

Lemma (Relation to Coordinate Matrices)**.** Let $T : V \to V$ be linear with $\dim V = n$ and let $A \in \text{Mat}_{n\times n}(\mathbb{F})$. Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V.

1. $\vec{v} \in V$ is an eigenvector for T with eigenvalue λ if and only if $[\vec{v}]_\beta$ is an eigenvector for $[T]_\beta$ with eigenvalue λ.
2. $\vec{v} \in \mathbb{F}^n$ is an eigenvector for A with eigenvalue λ if and only if $\vec{v} \in \mathbb{F}^n$ is an eigenvector for L_A with eigenvalue λ.

Theorem (Calculating Eigenvalues)**.** Suppose $\dim V = n$ and β is any basis for V. Then λ is an eigenvalue for $T : V \to V$ if and only if

$$\det(\lambda I_n - [T]_\beta) = 0.$$

The eigenspace E_λ is given by

$$E_\lambda = \ker(\lambda \mathrm{id}_V - T).$$

Definition (Characteristic Polynomial)**.** Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$, the *characteristic polynomial* of A is

$$f_A(t) = \det(tI_n - A).$$

Given $T : V \to V$, V finite-dimensional, the characteristic polynomial of T is

$$f_T(t) := f_{[T]_\beta}(t) \quad \text{for any basis } \beta \text{ of } V.$$

Definition (Algebraic and Geometric Multiplicities)**.** Suppose

$$f_T(t) = \prod_{i=1}^{r} (t - \lambda_i)^{n_i} g(t),$$

where:

- $\lambda_i \in \mathbb{F}$,
- $\lambda_i \neq \lambda_j$ for $i \neq j$,
- $g(t)$ has no roots in $\mathbb{F}$.

Note: $\lambda_1, \ldots, \lambda_r$ are exactly the eigenvalues of T.

Definition (Multiplicities)**.** The *geometric multiplicity* of an eigenvalue λ of $T : V \to V$ is $\dim(E_\lambda)$.

The *algebraic multiplicity* of $\lambda_i, 1 \leq i \leq r$, is n_i, as in the above definition. This is the maximal power of $(t - \lambda_i)$ dividing $f_T(t)$.

Definition (Eigenbasis)**.** Let $T : V \to V$ and let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. If all $\vec{v}_i$ are eigenvectors for T, we call β an *eigenbasis.*

By definition, $\forall i \in \{1, \ldots, n\}$, $\exists \lambda_i \in \mathbb{F}$ such that $T(\vec{v}_i) = \lambda_i \vec{v}_i$.

Theorem (Eigenbasis and Diagonal Matrix)**.** In summary, we've proved: β is an eigenbasis if and only if $[T]_\beta$ is diagonal.

Definition (Diagonalizable)**.** A transformation $T : V \to V$ is *diagonalizable* if there is a basis β for V such that $[T]_\beta$ is a diagonal matrix.

A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ is *diagonalizable* if $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is diagonalizable.

Definition (Splits)**.** A polynomial $f(t) = a_0 + a_1 t + \ldots + a_n t^n$ with $a_i \in \mathbb{F}$ *splits* over $\mathbb{F}$ if $f(t)$ factors into linear terms with roots in $\mathbb{F}$:

$$f(t) = (t - \lambda_1)^{n_1} \cdots (t - \lambda_r)^{n_r},$$

where $\lambda_i \in \mathbb{F}$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Field matters!

Lemma (Diagonalizable Equivalent Conditions)**.** Let V be finite-dimensional. Let $T : V \to V$ be linear and $A \in \text{Mat}_{n\times n}(\mathbb{F})$.

1. Let γ be a basis for V. T is diagonalizable if and only if $[T]_\gamma$ is diagonalizable.
2. A is diagonalizable if and only if A is similar to a diagonal matrix.

Theorem (More equivalent conditions to Diagonalizable)**.** T is diagonalizable $\Leftrightarrow$

1. $f_T(t)$ splits over $\mathbb{F}$,
2. For all eigenvalues of T, the algebraic multiplicity equals the geometric multiplicity.

Definition (Direct Sum)**.** Let $W_1, \ldots, W_r \subseteq V$ be subspaces. $W_1 + \cdots + W_r$ is a *direct sum*, written as $W_1 \oplus \cdots \oplus W_r$, if for all $w \in W_1 + \cdots + W_r$, the representation $w = u_1 + \cdots + u_r$ with $u_i \in W_i$ is unique.

That is, if $u_1 + \cdots + u_r = u_1' + \cdots + u_r'$ for $u_i, u_i' \in W_i$, then $u_i = u_i'$ for all i.

Lemma (A)**.** Let $T : V \to V$ and $\dim V = n$. Then $f_T(t)$ has degree n.

Let $\lambda \in \mathbb{F}$ be an eigenvalue for $T : V \to V$. Then:

$$1 \leq \text{Geometric multiplicity of } \lambda \leq \text{Algebraic multiplicity of } \lambda.$$

Lemma (B)**.** Let $\lambda_1, \ldots, \lambda_r$ be eigenvalues of $T : V \to V$. Then the sum $E_{\lambda_1} + \cdots + E_{\lambda_r}$ is direct.

Lemma (C)**.** Let V be finite-dimensional over F, $T : V \to V$ with $\lambda_1, \ldots, \lambda_r$ distinct roots of $f_T(t)$. Then T has a basis of eigenvectors for T if and only if $V = E_{\lambda_1} + \cdots + E_{\lambda_r}$.

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1 Inner Product Spaces

Definition (Inner Product)**.** Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. An *inner product* on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

(written as $\langle x, y \rangle$ for $x, y \in V$) such that for all $x, y, z \in V$ and all $c \in \mathbb{F}$:

(a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$ (Linearity in the first variable)

(b) $\langle cx, y \rangle = c\langle x, y \rangle$

(c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Conjugate symmetry)

(d) If $x \neq \vec{0}$, then $\langle x, x \rangle > 0$ (Positive-definiteness)

Theorem (IP Properties)**.** Let V be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then for all $x, y, z \in V$ and all $c \in \mathbb{F}$:

(a) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(b) $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$

(c) $\langle x, \vec{0}\rangle = \langle \vec{0}, x\rangle = 0$

(d) $\langle x, x\rangle = 0 \iff x = \vec{0}$

(e) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

Theorem (IP Properties Continued)**.** If V is an inner product space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, then:

(a) For all $x \in V$ and $c \in \mathbb{F}$:

$$\|cx\| = |c|\|x\|$$

where $c = a + ib \in \mathbb{C}$ and $|c| = \sqrt{a^2 + b^2}$.

(b) $\|x\| = 0 \iff x = \vec{0}$.

(c) If $x \neq \vec{0}$, then $\frac{x}{\|x\|}$ is normal.

Theorem (Orthogonal Projections)**.** Let V be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthogonal basis. Given $x \in V$,

$$x = \sum_{i=1}^{n} \frac{\langle x, \vec{v}_i\rangle}{\|\vec{v}_i\|^2}\vec{v}_i.$$

Corollary (Orthonormal Basis)**.** If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis (ONB) for V, then for all $x \in V$,

$$x = \sum_{i=1}^{n} \langle x, \vec{v}_i\rangle \vec{v}_i.$$

Definition (Orthogonal Projection)**.** Let V be an inner product space. Let $U \subseteq V$ be a subspace. Given an orthonormal basis (ONB) for U, $\{\vec{u}_1, \ldots, \vec{u}_k\} \subseteq U$, the orthogonal projection onto U is a function

$$P_U : V \to V$$

defined by

$$P_U(v) = \sum_{j=1}^{k} \frac{\langle v, \vec{u}_j\rangle}{\|\vec{u}_j\|^2}\vec{u}_j.$$

Proposition (Minimum value for projection)**.** Suppose V is finite-dimensional. Let U be a subspace of V, and let $v \in V$. Then the value $\|v - \vec{x}\|$ is minimized when $\vec{x} = P_U(v)$, where P_U is the orthogonal projection onto U.

Lemma (Pythagorean Theorem)**.** Let $\vec{x} \in U$ and $v \in V$. Then

$$\|v - \vec{x}\|^2 = \|P_U(v) - \vec{x}\|^2 + \|P_U(v) - v\|^2.$$

This is the Pythagorean theorem.

Theorem (Idea behind Gram Schmidt)**.** Let V be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $S = \{s_1, \ldots, s_n\}$ be a linearly independent set in V. Let $U_i = \text{span}\{s_1, \ldots, s_i\}$. Define:

$$t_1 = s_1,$$

$$t_j = s_j - P_{U_{j-1}}(s_j).$$

Then the set $S' = \{t_1, \ldots, t_n\}$ is orthogonal, and

$$\text{span}(S) = \text{span}(S').$$

Furthermore, the set $S'' = \left\{ \frac{t_1}{\|t_1\|}, \ldots, \frac{t_n}{\|t_n\|} \right\}$ is an orthonormal basis (ONB) for span(S).

Theorem (Gram-Schmidt Algorithm)**.** Let $S = \{w_1, \ldots, w_n\}$ be a linearly independent set in V. Define:

$$v_1 = w_1,$$

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \quad \text{for } 2 \leq k \leq n.$$

Let $S' = \{v_1, \ldots, v_n\}$. Then S' is orthogonal. Furthermore, let

$$S'' = \left\{ \frac{v_1}{\|v_1\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}.$$

Then S'' is orthonormal.

Definition (Adjoint Operator)**.** Given $T : V \to V$, let $T^* : V \to V$ be determined by $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$.

Theorem (On Adjoint Operators)**.** Let $T : V \to V$ be a linear operator, and let β be an orthonormal basis (ONB) for V. Then,

$$[T^*]_\beta = [T]^*_\beta$$

where for $A \in M_{n \times n}(F)$, $(A^*)_{ij} = \overline{A_{ji}}$.

Definition (Normal and Self-adjoint)**.** We say that $T : V \to V$ is **normal** if $TT^* = T^*T$.
We say that $T : V \to V$ is **self-adjoint** if $T^* = T$.
We say that $A \in M_{n \times n}(F)$ is **normal** if $AA^* = A^*A$.
We say that $A \in M_{n \times n}(F)$ is **self-adjoint** if $A^* = A$.

Theorem ($\star$ - Stronger diagonalization)**.** Let $T : V \to V$ be linear.

- If $F = \mathbb{C}$, T is normal $\iff$ there exists an orthonormal basis (ONB) for V of eigenvectors for T.
- If $F = \mathbb{R}$, T is self-adjoint $\iff$ there exists an ONB for V of eigenvectors for T.

Theorem (Facts about Adjoints)**.** Let $T : V \to V$ be normal. Then:

1. $\|T(x)\| = \|T^*(x)\| \, \forall x \in V$.
2. $T - cI$ is normal $\forall c \in \mathbb{F}$.
3. If x is an eigenvector for T with eigenvalue λ, then x is an eigenvector for T^* with eigenvalue $\overline{\lambda}$.
4. If $\lambda_1 \neq \lambda_2$, $x, y \in V$, $T(x) = \lambda_1 x$, $T(y) = \lambda_2 y$, then $\langle x, y \rangle = 0$. i.e. Eigenvectors for distinct eigenvalues are orthogonal.

Theorem (Adjoints and Kernels and Images)**.** Let $T : V \to V$ be a linear operator on a finite-dimensional inner product space V, and let T^* be the adjoint of T. Then the following properties hold:

1. $\text{Im}(T^*)^\perp = \ker(T)$
2. If V is finite-dimensional, $\text{Im}(T^*) = \ker(T)^\perp$
3. $\text{Im}(T)^\perp = \ker(T^*)$
4. If V is finite-dimensional, $\text{Im}(T) = \ker(T^*)^\perp$

Theorem (Schur's)**.** Suppose that $T : V \to V$ is linear and $f_T(t)$ splits. Then there exists an orthonormal basis (ONB) for V such that $[T]_\beta$ is upper-triangular.

Theorem (Fundamental Theorem of Algebra)**.** Every non-constant polynomial $g(t) \in \text{Poly}(\mathbb{C})$ has a root in $\mathbb{C}$.

Theorem (Spectral Theorem)**.** Let V be a finite-dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
Let $T : V \to V$ be linear and normal if $\mathbb{F} = \mathbb{C}$, self-adjoint if $\mathbb{F} = \mathbb{R}$.
Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T, and let $W_i = E_{\lambda_i}$.
Then:

1. $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$
2. $W_i \perp \sum_{\substack{j=1 \\ j \neq i}}^{k} W_j$

   Let $T_i = P_{W_i}$ for $1 \leq i \leq k$:

3. $I = T_1 + \cdots + T_k$
4. $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$

What does this mean?
(4): Decompose T into scaled projection operators.
The spectral decomposition of T.